Eikonal Approximation

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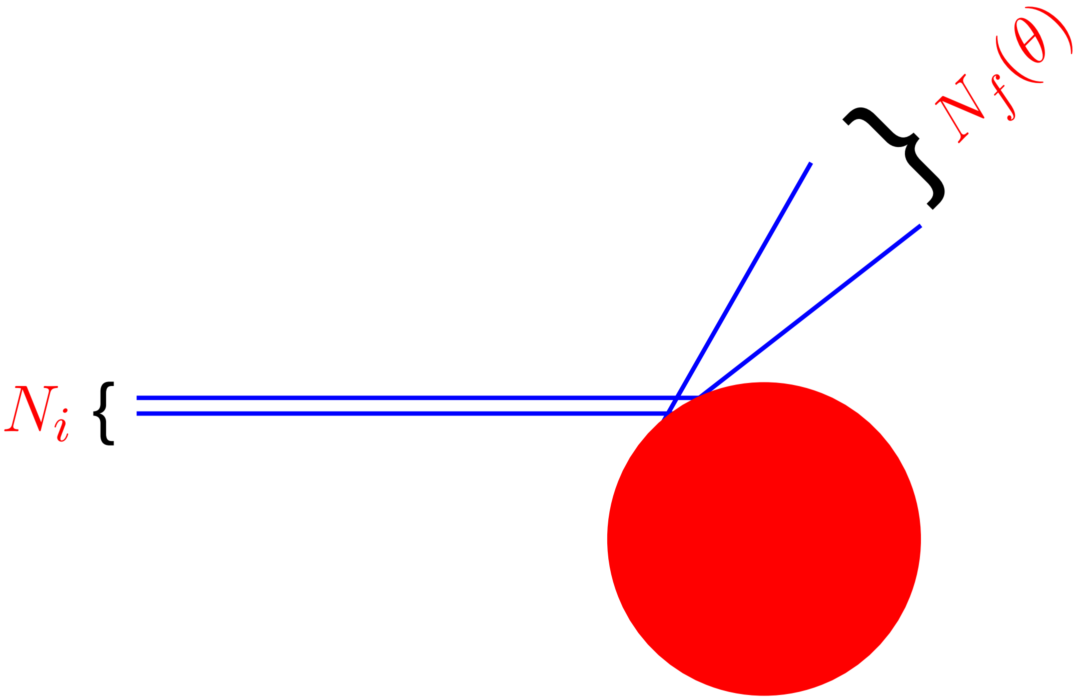
I wish I could be an eikonal. I could then move along a straight line.

Classical mechanics



Quantum mechanics

Schroedinger equation:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r},t)\right]\psi'(\vec{r},t) = -i\hbar\frac{\partial}{\partial t}\psi'(\vec{r},t)$$

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$$\longrightarrow \vec{k}$$

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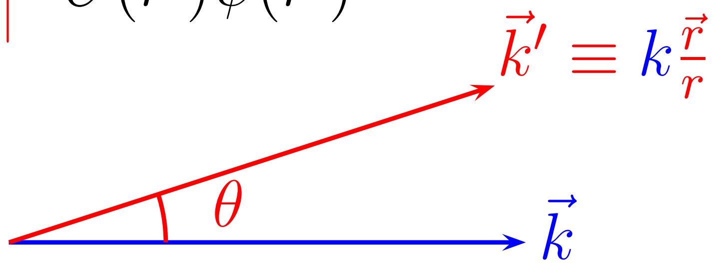
$$\psi_{r\to\infty}(\vec{r}) = A_0\, e^{i\vec{k}\cdot\vec{r}} + f(\theta,\phi)\;\frac{e^{ikr}}{r}$$



$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\; e^{-ik\frac{\vec{r}}{r}\cdot\vec{r}\,'}U(\vec{r}\,')\psi(\vec{r}\,')$$

$$= -\frac{1}{4\pi}\int d^3r'\; e^{-i\vec{k}\,'\cdot\vec{r}\,'}U(\vec{r}\,')\psi(\vec{r}\,')$$

Eikonal approximation

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\; e^{-i\vec{k}'\cdot\vec{r}'}U(\vec{r}')\psi(\vec{r}')$$

$$\left[\nabla^2 + k^2\right]\psi(\vec{r}) = U(\vec{r})\psi(\vec{r})$$

Eikonal approximation

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\; e^{-i\vec{k}'\cdot\vec{r}'}U(\vec{r}')\psi(\vec{r}')$$

$$\left[\nabla^2 + k^2\right]\psi(\vec{r}) = U(\vec{r})\psi(\vec{r})$$

Conditions for the validity of eikonal approximation are:

- $\frac{V}{E} \ll 1, \;\; \theta \ll 1$
- $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$

Eikonal approximation

$$f(\theta, \phi) = -\frac{1}{4\pi} \int d^3r' \; e^{-i\vec{k}' \cdot \vec{r}'} U(\vec{r}') \psi(\vec{r}')$$

$$\left[\nabla^2 + k^2\right] \psi(\vec{r}) = U(\vec{r}) \psi(\vec{r})$$

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$$\left[\nabla^2 + k^2\right]\psi(\vec{r}) = U(\vec{r})\psi(\vec{r})$$

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\phi(\vec{r})$$

$$\vec{r} \equiv (\vec{b}, z)$$

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$$= e^{i\vec{k}\cdot\vec{r}}\;\; ?$$

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$$\left[2ik\frac{\partial}{\partial z} - U(\vec{b},z)\right]\phi(\vec{b},z) = -\nabla^2\phi(\vec{b},z)$$

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Eikonal approximation

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$$\phi(\vec{b}, z) = \eta(\vec{b}, z) - \int d^2b' \int_{-\infty}^{\infty} dz'\; G_e(\vec{b}, z, \vec{b}', z')\; \nabla'^2\phi(\vec{b}', z')$$

- $\frac{V}{E} \ll 1$ and $\theta \ll 1$
- $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$

Eikonal approximation

$$f(\theta,\phi) = -\frac{1}{4\pi}\int d^3r'\; e^{-i\vec{k}'\cdot\vec{r}'}U(\vec{r}')\psi(\vec{r}')$$

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$$\left[2ik\frac{\partial}{\partial z} - U(\vec{b}, z)\right] \eta(\vec{b}, z) = 0$$

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$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}\phi(\vec{r}) \qquad\qquad \vec{r} \equiv (\vec{b},z)$$

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$$= \quad ?$$

$$-i\vec{k}'\cdot\vec{r}' + i\vec{k}\cdot\vec{r}' \approx -ikb'\theta\cos(\phi-\phi')$$

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$$= -\frac{1}{4\pi}\int d^2b'\; e^{-ikb'\theta\cos(\phi-\phi')} \int \underbrace{dz' U(\vec{b}', z')}_{dt}\; e^{\underbrace{\frac{1}{2ik}\int_{-\infty}^{z'} du\; U(\vec{b}',u)}_{t}}$$

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$$= \frac{k}{2\pi i}\int_0^{\infty} b'db' \int_0^{2\pi} d\phi'\; e^{-ikb'\theta\cos(\phi-\phi')}\left[e^{i\chi(\vec{b}')} - 1\right]$$

$$\chi(\vec{b}') = -\frac{k}{2}\frac{1}{E}\int_{-\infty}^{+\infty} dz'\; V(\vec{b}',z')$$

- $\frac{V}{E} \ll 1$ and $\theta \ll 1$
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Eikonal Approximation

$$f(\theta,\phi) = \frac{k}{2\pi i}\int_0^\infty b'db' \int_0^{2\pi} d\phi' \, e^{-ikb'\theta\cos(\phi-\phi')}\left[e^{i\chi(\vec{b}')} - 1\right]$$

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Eikonal Approximation

$$f(\theta,\phi) = \frac{k}{2\pi i}\int_0^\infty b' db' \int_0^{2\pi} d\phi' \; e^{-ikb'\theta\cos(\phi-\phi')}\left[e^{i\chi(\vec{b}')} - 1\right]$$

For ϕ independent potentials we have

$$z = ka$$
$$b = at$$

$$f(\theta) = a\,\frac{1}{i}\,ka\int_0^\infty t\;dt\;J_0(t\;ka\theta)\left[e^{ika\frac{V}{E}\xi(t)} - 1\right]$$

$$\xi(t) = -\frac{1}{2}\frac{1}{V}\int_{-\infty}^{+\infty} du\;V(at,au)$$

$$\chi(\vec{b}') = -\frac{k}{2}\frac{1}{E}\int_{-\infty}^{+\infty} dz'\;V(\vec{b}',z')$$

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For ϕ independent potentials we have

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$$f(\theta) = a\;\frac{1}{i}\; ka \int_0^\infty t\; dt\;\; J_0(t\; ka\theta)\left[e^{ika\frac{V}{E}\xi(t)} - 1\right]$$

$$\sigma_{\text{scatt.}} = \int d\Omega \mid f(\theta,\phi)\mid^2$$

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8\int_0^\infty t\; dt\;\; \sin^2\left[ka\frac{V}{E}\;\xi(t)\right]$$

$$\xi(t) = -\frac{1}{2}\frac{1}{V}\int_{-\infty}^{+\infty} du\; V(at, au)$$

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Spherical well potential

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8\int_0^\infty t\ dt\ \sin^2\left[ka\frac{V}{E}\ \xi(t)\right]$$

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Spherical well potential

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$$V(\vec{r}) = \begin{cases} V & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a \end{cases}$$

- $\frac{V}{E} \ll 1$ and $\theta \ll 1$
- $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$

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$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8 \int_0^\infty t \, dt \;\; \sin^2 \left[ka \frac{V}{E} \; \xi(t)\right]$$

$$\xi(t) = -\frac{1}{2}\frac{1}{V} \int_{-\infty}^{+\infty} du \, V(at, au)$$

$$= -\sqrt{1 - t^2}$$

$$V(\vec{r}) = \begin{cases} V & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a \end{cases}$$

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$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8\int_0^1 t\ dt\ \sin^2\left[ka\frac{V}{E}\ \sqrt{1-t^2}\right]$$

$$1 \ll ka\frac{V}{E} \ll \sqrt{ka}$$

$$V(\vec{r}) = \begin{cases} V & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a \end{cases}$$

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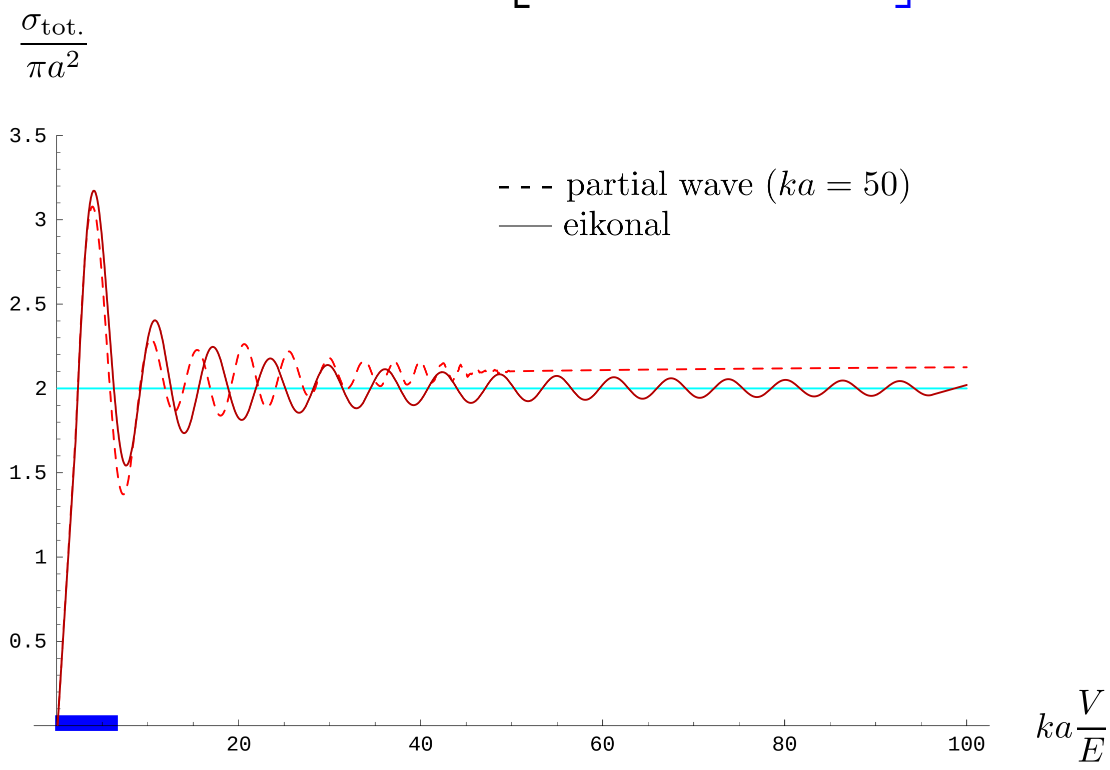


$$ka = 50$$

$$1 \ll ka\frac{V}{E} \ll 7$$

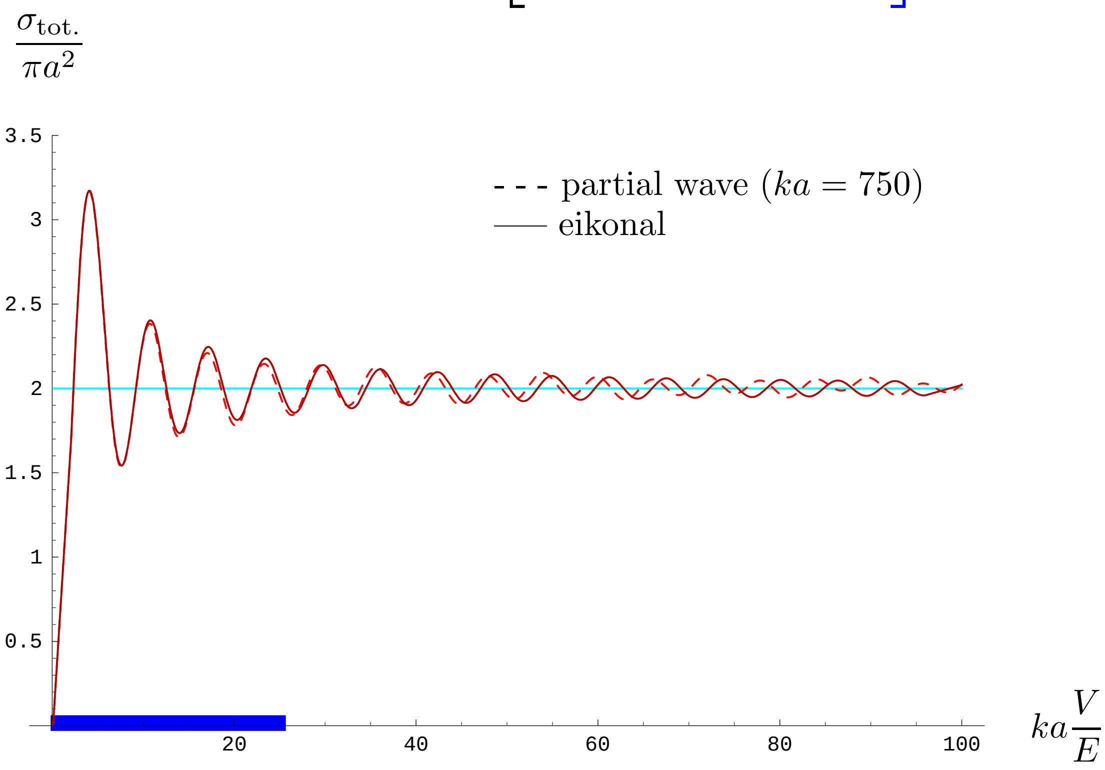
$$V(\vec{r}) = \begin{cases} V & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a \end{cases}$$

- $\frac{V}{E} \ll 1$ and $\theta \ll 1$
- $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$

Spherical well potential

$$\frac{\sigma_{\text{scatt.}}}{\pi a^2} = 8 \int_0^1 t\ dt\ \sin^2\left[ka\frac{V}{E}\sqrt{1-t^2}\right]$$

$$1 \ll ka\frac{V}{E} \ll \sqrt{ka}$$



$$ka = 750$$

$$1 \ll ka\frac{V}{E} \ll 27$$

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Yukawa potential

$$f(\theta) = a\,\frac{1}{i}\,ka\int_0^\infty t\,dt\;J_0(t\,ka\theta)\left[e^{ika\frac{V}{E}\xi(t)} - 1\right]$$

$$\xi(t) = -\frac{1}{2}\frac{1}{V}\int_{-\infty}^{+\infty} du\;V(at, au)$$

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$$V = \frac{e^2}{a}$$

$$ka\frac{V}{E} = \frac{2e^2}{\hbar v}$$

$$\hbar k = p = mv$$

$$V(r) = -V\;\frac{e^{-r/a}}{r/a}$$

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$$= \int_0^\infty d\theta\,e^{-t\cosh\theta}$$

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$$K_0(t) \approx \ln\frac{2}{t} - \gamma, \quad \gamma = 0.577\ldots$$

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$$\mid f(\theta)\mid_{\theta\neq 0} = \frac{1}{2k\left(\frac{\theta}{2}\right)^2}\,\frac{2}{ka\frac{V}{E}}\,\frac{\Gamma\left(1-i\frac{2}{ka\frac{V}{E}}\right)}{\Gamma\left(1+i\frac{2}{ka\frac{V}{E}}\right)}$$

$$V(r) = -V\,\frac{e^{-r/a}}{r/a}$$

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$$\sin\frac{\theta}{2} \approx \frac{\theta}{2}$$

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Quantum electrodynamics

$$Z[\mathcal{J}] = \langle 0_+ \mid 0_- \rangle^{\mathcal{J}}$$

$$\mathcal{J}(x) \equiv \{J_\mu(x), \eta(x), \bar{\eta}(x)\}$$

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$$-(-i)\frac{\delta Z[\mathcal{J}]}{\delta\eta(x)}\left[-i\gamma^\mu\overleftarrow{\partial}_\mu - m + e\gamma^\mu(-i)\frac{\overleftarrow{\delta}}{\delta J^\mu(x)}\right] = \bar{\eta}(x)\ Z[\mathcal{J}]$$

$$\left[g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right](-i)\frac{\delta Z[\mathcal{J}]}{\delta J^\mu(x)} = J^\mu(x)\ Z[\mathcal{J}] + e\ (-i)\frac{\delta}{\delta\eta(x)}\ \gamma^\mu\ (-i)\frac{\delta Z[\mathcal{J}]}{\delta\bar{\eta}(x)}$$

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$$(-i)\frac{\delta Z[\mathcal{J}]}{\delta J^\mu(x)} = \langle 0_+ \mid A_\mu(x) \mid 0_- \rangle^{\mathcal{J}}$$

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$$Z[\mathcal{J}] = e^{\frac{i}{2}\int\int J_\mu D_+ J^\mu} \left\{ e^{\frac{i}{2}\int\int \frac{\delta}{\delta A^\mu} D_+ \frac{\delta}{\delta A_\mu}} \; e^{i\int\int \bar{\eta} G_+[A]\eta} \; e^{L[A]} \right\}_{A=\int DJ}$$

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$$-\partial^2 D_+ = \delta^{(4)} \quad \text{and} \quad -(\gamma^\mu \partial_\mu - m) S_+ = \delta^{(4)}$$

$$L[A] = -\mathrm{Tr}\ln G_+[A] + \mathrm{Tr}\ln S_+$$

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Solution to quantum electrodynamics boils down to solving a differential equation for $G_+[A]$

$$\left[-(i\gamma^\mu\partial_\mu - m) + e\gamma^\mu A_\mu(x)\right] G_+[x, x'; A^\mu] = \delta^{(4)}(x - x')$$

Electron-electron scattering

$$Z[\mathcal{J}] = e^{\frac{i}{2}\int\int J_\mu D_+ J^\mu} \left\{ e^{\frac{i}{2}\int\int \frac{\delta}{\delta A^\mu} D_+ \frac{\delta}{\delta A_\mu}} \; e^{i\int\int \bar{\eta} G_+[A] \eta} \; e^{L[A]} \right\}_{A = \int DJ}$$

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$$Z[\mathcal{J}] = Z[0] + \ldots + T + \ldots$$

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$$T = \frac{1}{4}\int\int\int\int \; \bar{\eta}(x_2)\bar{\eta}(x_1)\eta(y_2)\eta(y_1) \; G(x_1, y_1, x_2, y_2)$$

$$G(x_1, y_1, x_2, y_2) = \left\{ (-i)^4 \frac{\delta}{\delta\bar{\eta}_A(x_2)} \frac{\delta}{\delta\bar{\eta}_B(x_1)} \frac{\delta}{\delta\eta_A(y_2)} \frac{\delta}{\delta\eta_A(y_1)} Z[\mathcal{J}] \right\}_{\mathcal{J}=0}$$

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$$\bar{\eta}(x_1) = e^{-ip_1x_1}\bar{u}(p_1)[\gamma^\mu p_{1_\mu} + m]$$

$$\eta(y_1) = e^{ip'_1y_1}[\gamma^\mu p'_{1_\mu} + m]u(p'_1)$$

$$\bar{\eta}(x_2) = e^{-ip_2x_2}\bar{u}(p_2)[\gamma^\mu p_{2_\mu} + m]$$

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Eikonal approximation

$$Z[\mathcal{J}] = e^{\frac{i}{2}\int\int J_\mu D_+ J^\mu} \left\{ e^{\frac{i}{2}\int\int \frac{\delta}{\delta A^\mu} D_+ \frac{\delta}{\delta A_\mu}} \; e^{i\int\int \bar{\eta} G_+[A]\eta} \; \cancel{e^{L[A]}} \right\}_{A = \int DJ}$$

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$$\begin{aligned} T = -i\int_0^1 d\lambda\, e^{\lambda \frac{i}{2}\int\int \frac{\delta}{\delta A} D_+ \frac{\delta}{\delta A}} \int d^4 z_1 \int d^4 z_2 \; D_+^{\mu\nu}(z_1 - z_2) \\ \times \int d^4 x_1 e^{-ip_1 x_1} \bar{u}(p_1)[\gamma^\mu p_{1\mu} + m] G_+[x_1, z_1; A_1] \\ \times \; e\gamma_\mu \int d^4 y_1 e^{ip_1' y_1} G_+[z_1, y_1; A_1][\gamma^\mu p'_{1\mu} + m] u(p_1') \\ \times \int d^4 x_2 e^{-ip_2 x_2} \bar{u}(p_2)[\gamma^\mu p_{2\mu} + m] G_+[x_2, z_2; A_2] \\ \times \; e\gamma_\mu \int d^4 y_2 e^{ip_2' y_2} G_+[z_2, y_2; A_2][\gamma^\mu p'_{2\mu} + m] u(p_2') \end{aligned}$$

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$$\times \int d^4 x_1 e^{-ip_1 x_1} \bar{u}(p_1)[\gamma^\mu p_{1_\mu} + m] G_+[x_1, z_1; A_1]$$

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\end{aligned}
$$

$$
G_+[x, y; A] \rightarrow i \int_0^\infty ds\; e^{-ims} \delta(x - y - s\tfrac{p}{m})\; e^{ie \int_0^s d\tau \frac{1}{m} p^\mu A_\mu (x - \tau \frac{p}{m})}
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$$
T(p_1, p_1', p_2, p_2') = (2\pi)^4 \delta^{(4)}(p_1 + p_1' - p_2 - p_2') M(p_1, p_1', p_2, p_2')
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$$\mid M(p_1, p_1', p_2, p_2') \mid = \frac{e^2}{2m^2}\frac{(p_1+p_2)^2}{(p_1-p_1')^2}\frac{\Gamma(1-i\frac{e^2}{4\pi})}{\Gamma(1+i\frac{e^2}{4\pi})}$$

Conclusion and things to do

- Eikonal approximation in quantum mechanics works for $\frac{V}{E} \ll 1$, $\theta \ll 1$, and $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$.

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- Eikonal approximation in quantum mechanics works for $\frac{V}{E} \ll 1$, $\theta \ll 1$, and $\frac{1}{V/E} \ll ka \ll \frac{1}{(V/E)^2}$.
- Is eikonal approximation valid for singular potentials? It seems to work for Yukawa potential? Will it work for Coulomb potential?
- To understand eikonal approximation in quantum field theories.