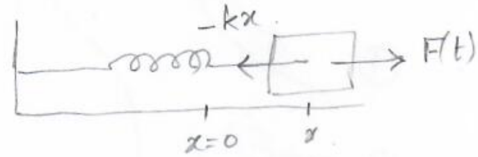
Date: 2024 Feb 17



1 $$m a = -kx + F(t)$$

$$\left(m \frac{d^2}{dt^2} + k\right) x(t) = F(t)$$

2 Action: $$W[x;F] = \int_{t_1}^{t_2} dt \; L\left(x(t), \frac{d}{dt}x(t), t\right)$$

$$W[x;F] = \int_{t_1}^{t_2} dt \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - \frac{1}{2} k x^2 + F x\right]$$

Stationary action principle.

$$\frac{\delta W}{\delta x(t)} = 0 \Rightarrow 1$$

$$\frac{\delta W}{\delta x(t)} = -m \frac{d^2}{dt^2} x - kx + F$$

$$\frac{\delta W}{\delta F(t)} = x(t)$$

3 Transition amplitude:

$$Z[F] = \langle \, | \, \rangle^F$$

4 Schwinger's quantum action principle.

$$\delta Z[F] = \frac{i}{\hbar} \langle \, | \, \delta W \, | \, \rangle^F$$

(5)
$$\frac{\delta Z[F]}{\delta x(t)} = \frac{i}{\hbar} \langle 0 | \left(-m \frac{d^2}{dt^2}x - kx + F \right) | \rangle$$

$$\frac{\delta Z}{\delta x(t)} = 0 \quad \Rightarrow \quad \left(m \frac{d^2}{dt^2} + k \right) \langle | x(t) | \rangle^F = F(t) \langle | \rangle^F$$

(6)
$$\frac{\delta Z[F]}{\delta F(t)} = \frac{i}{\hbar} \langle | x(t) | \rangle^F$$

$$\langle | x(t) | \rangle^F = \frac{\hbar}{i} \frac{\delta Z[F]}{\delta F(t)}$$

(7) Define:
$$\langle x(t) \rangle = \frac{\langle | x(t) | \rangle^F}{\langle | \rangle}$$

$$= \frac{1}{Z[F]} \frac{\hbar}{i} \frac{\delta Z[F]}{\delta F(t)}$$

(8)
$$\left(m \frac{d^2}{dt^2} + k \right) \langle x(t) \rangle = F(t)$$

$$\left(m \frac{d^2}{dt^2} + k \right) G(t, t') = \delta(t - t')$$

$$\langle x(t) \rangle = \int_{-\infty}^{+\infty} dt' \, G(t, t') \, F(t')$$

(9) Using (8)

$$\frac{\delta \langle x(t) \rangle}{\delta F(t')} = G(t,t')$$

$$G(t,t') = \frac{\delta}{\delta F(t')} \frac{1}{Z[F]} \frac{\hbar}{i} \frac{\delta Z[F]}{\delta F(t)}$$

$$G(t,t') = \frac{1}{Z[F]} \frac{\hbar}{i} \frac{\delta}{\delta F(t')} \frac{\delta Z}{\delta F(t)} - \frac{1}{Z^2} \frac{\delta Z}{\delta F(t')} \frac{\hbar}{i} \frac{\delta Z}{\delta F(t)}$$

$$\frac{\hbar}{i} G(t,t') = \frac{1}{Z[F]} \frac{\hbar}{i} \frac{\delta}{\delta F(t')} \frac{\hbar}{i} \frac{\delta Z}{\delta F(t)} - \frac{1}{Z} \frac{\hbar}{i} \frac{\delta Z}{\delta F(t')} \frac{1}{Z} \frac{\hbar}{i} \frac{\delta Z}{\delta F(t)}$$

(10) Using (8)

$$\frac{\hbar}{i} \frac{\delta Z}{\delta F(t)} = Z[F] \int_{-\infty}^{+\infty} dt' \, G(t,t') \, F(t')$$

$$Z[F] = Z[0] \, e^{\frac{i}{\hbar} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt' \, F(t) \, G(t,t') \, F(t')}$$

(11)
$$\frac{\hbar}{i}\frac{\delta Z}{\delta F(t)} = Z[F]\int_{-\infty}^{+\infty} dt'\, G(t,t')\, F(t')$$

$$\frac{\hbar}{i}\frac{\delta}{\delta F(t')}\frac{\hbar}{i}\frac{\delta Z}{F(t)} = Z[F]\frac{\hbar}{i}G(t,t') + Z[F]\int_{-\infty}^{+\infty} d\bar{t}'\, G(t',\bar{t}')F(\bar{t}')\int_{-\infty}^{+\infty} d\bar{t}''\, G(t,\bar{t})F.$$

$$\frac{1}{Z}\frac{\hbar}{i}\frac{\delta}{\delta F(t')}\frac{\hbar}{i}\frac{\delta Z}{F(t)} = \frac{\hbar}{i}G(t,t') + \langle x(t)\rangle\langle x(t')\rangle$$

(12) Define:
$$\langle x(t)\, x(t')\rangle = \frac{1}{Z[F]}\frac{\hbar}{i}\frac{\delta}{\delta F(t')}\frac{\hbar}{i}\frac{\delta}{\delta F(t)} Z[F].$$

(13) Using (12) in (9)
$$\frac{\hbar}{i}G(t,t') = \langle x(t)\, x(t')\rangle^F - \langle x(t)\rangle^F\langle x(t')\rangle^F$$

(14) Quantum vacuum fluctuations:
$$\langle x(t)\rangle^{F=0} = 0$$
$$\langle x(t)\, x(t')\rangle^{F=0} = \frac{\hbar}{i}G(t,t')$$

(15) Zero point energy

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} k x^2 - F(t)\, x(t)$$

$$= \lim_{t \to t'} \frac{1}{2} m \frac{d x(t')}{dt'} \frac{d x(t)}{dt} + \frac{1}{2} k\, x(t')\, x(t) - F(t)\, x(t)$$

$$\langle E \rangle = \lim_{t \to t'} \left[\frac{1}{2} m \frac{\partial}{\partial t} \frac{\partial}{\partial t'} + \frac{1}{2} k \right] \langle x(t)\, x(t') \rangle - F(t) \langle x(t) \rangle$$

$$\langle E \rangle^{F=0} = \lim_{t \to t'} \left[\frac{1}{2} m \frac{\partial}{\partial t} \frac{\partial}{\partial t'} + \frac{1}{2} k \right] \frac{\hbar}{i} G(t,t')$$

$$= \lim_{t \to t'} \left(\frac{\partial}{\partial t} \frac{\partial}{\partial t'} + \omega_0^2 \right) \frac{1}{2} m \frac{\hbar}{i} G(t,t')$$

$\omega_0^2 = \frac{k}{m}$

(16)
$$\left(m \frac{\partial^2}{\partial t^2} + k \right) G(t,t') = \delta(t-t')$$

$$\left(\frac{\partial^2}{\partial t^2} + \omega_0^2 \right) m\, G(t,t') = \delta(t-t')$$

(17)
$$m\, G(t,t') = \begin{cases} A\, e^{i\omega_0 t} + B\, e^{-i\omega_0 t} & t < t' \\ C\, e^{i\omega_0 t} + D\, e^{-i\omega_0 t} & t' < t \end{cases}$$

(18) Continuity conditions

$$G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta} = 0$$

$$\frac{\partial}{\partial t} G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta} = 1$$

(19)
$$(C-A)\, e^{i\omega_0 t'} + (D-B)\, e^{-i\omega_0 t'} = 0$$

$$(C-A)\, e^{i\omega_0 t'} - (D-B)\, e^{-i\omega_0 t'} = \frac{1}{i\omega_0}.$$

$$C - A = \frac{e^{-i\omega_0 t'}}{2i\omega_0}$$

$$D - B = -\frac{e^{i\omega_0 t'}}{2i\omega_0}$$

(20)
$$m\, G(t,t') = A e^{i\omega_0 t} + B e^{-i\omega_0 t} + \left(\frac{e^{-i\omega_0 t'}}{2i\omega_0} e^{i\omega_0 t} - \frac{e^{i\omega_0 t'}}{2i\omega_0} e^{-i\omega_0 t} \right) \theta(t-t')$$

$$m\, G(t,t') = \begin{matrix}\text{correlation}\\ \text{free solution}\end{matrix} + \frac{1}{\omega_0}\, \theta(t-t')\, \sin \omega_0 (t-t')$$

→ for now - drop.

(21) $$\frac{\partial}{\partial t} m\, G(t,t') = \frac{1}{\omega_0}\, \delta(t-t')\, \sin\omega_0(t-t') + \theta(t-t')\, \cos\omega_0(t-t')$$

$$\frac{\partial}{\partial t'}\frac{\partial}{\partial t} m\, G(t,t') = \frac{1}{\omega_0}\left\{\frac{\partial}{\partial t'}\, \delta(t-t')\right\} \sin\omega_0(t-t') - \delta(t-t')\, \cos\omega_0(t-t')$$

$$+\, \delta(t-t')\, \cos\omega_0(t-t') + \omega_0\, \theta(t-t')\, \sin\omega_0(t-t')$$

$$\frac{\partial}{\partial t'}\frac{\partial}{\partial t} m\, G(t,t') = \frac{1}{\omega_0}\left\{\frac{\partial}{\partial t'}\, \delta(t-t')\right\} \sin\omega_0(t-t') + \omega_0\, \theta(t-t')\, \sin\omega_0(t-t')$$

$$\left(\frac{\partial}{\partial t'}\frac{\partial}{\partial t} + \omega_0^2\right) m\, G(t,t') = -(t-t')\left\{\frac{\partial}{\partial t}\, \delta(t-t')\right\} \frac{\sin\omega_0(t-t')}{\omega_0(t-t')}$$

$$+\, 2\omega_0\, \theta(t-t')\, \sin\omega_0(t-t')$$

$$\lim_{t\to t'}\left(\frac{\partial}{\partial t'}\frac{\partial}{\partial t} + \omega_0^2\right) m\, G(t,t') = \bar{\delta}(t-t')$$

where $$\bar{\delta}(t) = -t\, \frac{d}{dt}\, \delta(t)$$

(22) $$\langle E\rangle^{F=0} = \frac{1}{2}\, \frac{\hbar}{i}\, \bar{\delta}(t-t')$$?

$$\frac{1}{2}\hbar\omega_0$$

$$\langle 0_+|0_-\rangle^{F=0} = \langle 0_-|e^{-iH\tau}|0_-\rangle = e^{-iE\tau}$$