① Maxwell's equation:

(i) $\vec{\nabla}\cdot\vec{D}(\vec{r},\omega) = \cancel{\rho(\vec{r},\omega)} - \vec{\nabla}\cdot\vec{P}(\vec{r},\omega)$ $\quad \vec{D}(\vec{r},\omega) = \overleftrightarrow{\epsilon}(\vec{r},\omega)\cdot\vec{E}(\vec{r},\omega)$

$\vec{B}(\vec{r},\omega) = \overleftrightarrow{\mu}(\vec{r},\omega)\cdot\vec{H}(\vec{r},\omega)$

(ii) $\vec{\nabla}\cdot\vec{B}(\vec{r},\omega) = 0$

(iii) $-\vec{\nabla}\times\vec{E}(\vec{r},\omega) = i\omega\,\vec{B}(\vec{r},\omega)$

(iv) $-\vec{\nabla}\times\vec{H}(\vec{r},\omega) = -i\omega\,\vec{D}(\vec{r},\omega) + \cancel{\vec{j}(\vec{r},\omega)} - i\omega\,\vec{P}(\vec{r},\omega)$

② For $\rho = 0$ and $\vec{j} = 0$, (iii) and (iv) implies (i) and (ii).

③ Action

$$\frac{\text{energy}}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2 \rightarrow \frac{1}{2}\vec{E}\cdot\vec{D}$$

$$\int d^3r \int_{-\infty}^{+\infty} dt\, \frac{1}{2}\vec{E}(\vec{r},t)\cdot\vec{D}(\vec{r},t)$$

$$= \int d^3r \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dt'\, \frac{1}{2}\vec{E}(\vec{r},t)^*\cdot\overleftrightarrow{\epsilon}(\vec{r},t-t')\cdot\vec{E}(\vec{r},t')$$

$$= \int d^3r \int dt \int dt'\, \frac{1}{2}\int\frac{d\omega_1}{2\pi} e^{+i\omega_1 t}\vec{E}(\vec{r},\omega_1)^*\cdot\int\frac{d\omega_2}{2\pi} e^{-i\omega_2(t-t')}\overleftrightarrow{\epsilon}(\omega_2)\int\frac{d\omega_3}{2\pi}e^{-i\omega_3 t'}\vec{E}(\omega_3)$$

$$= \int d^3r \int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\int\frac{d\omega_3}{2\pi}\,\vec{E}(\vec{r},\omega_1)^*\cdot\overleftrightarrow{\epsilon}(\vec{r},\omega_2)\cdot\vec{E}(\vec{r},\omega_3)\; 2\pi\delta(\omega_1-\omega_2)\, 2\pi\delta(\omega_2-\omega_3)$$

$$= \int d^3r \int\frac{d\omega}{2\pi}\,\vec{E}(\vec{r},\omega)^*\cdot\overleftrightarrow{\epsilon}(\vec{r},\omega)\cdot\vec{E}(\vec{r},\omega)$$

$$=$$

(4) $$W[\vec{E}, \vec{H}, \phi, \vec{A}; \vec{P}] = \int d^3r \int \frac{d\omega}{2\pi} \, \mathcal{L}.$$

$$\mathcal{L} = -\frac{1}{2} \vec{E}^* \cdot \overleftrightarrow{\epsilon} \cdot \vec{E} + \vec{E}^* \cdot \overleftrightarrow{\epsilon} \cdot \left[-\vec{\nabla}\phi + i\omega\vec{A} \right]$$
$$+ \frac{1}{2} \vec{H}^* \cdot \overleftrightarrow{\mu} \cdot \vec{H} - \vec{H}^* \cdot \left(\vec{\nabla} \times \vec{A} \right)$$
$$- \left(\rho\phi - \vec{j} \cdot \vec{A} \right)$$
$$+ \vec{P}^* \cdot \left[-\vec{\nabla}\phi + i\omega\vec{A} \right]$$

(5) $$\delta\vec{E}: \quad 0 = -\overleftrightarrow{\epsilon} \cdot \vec{E} + \overleftrightarrow{\epsilon} \cdot \left[-\vec{\nabla}\phi + i\omega\vec{A} \right]$$
$$\Rightarrow \quad \vec{E} = -\vec{\nabla}\phi + i\omega\vec{A}$$

$$\delta\vec{H}: \quad 0 = \overleftrightarrow{\mu} \cdot \vec{H} - \vec{\nabla} \times \vec{A}$$
$$\Rightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$\delta\phi: \quad 0 = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\vec{E}(-\omega) \cdot \overleftrightarrow{\epsilon}(\omega) + \vec{P}(-\omega) \right] \left\{ -\vec{\nabla} \, \delta\phi(\omega) \right\} - \rho\,\delta\phi$$
$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, \delta\phi(\omega) \left(\vec{\nabla}_j \cdot \left[E_i(-\omega)\, \epsilon_{ij}(\omega) + P_j(-\omega) \right] - \rho \right)$$

$\epsilon_{ij}(\omega) = \epsilon_{ji}(-\omega)$ (see (6))

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, \delta\phi(\omega) \left(\vec{\nabla}_j \cdot \left[\epsilon_{ji}(-\omega)\, E_i(-\omega) + P_j(-\omega) \right] - \rho \right)$$

$$\omega \to -\omega$$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \, \delta\phi(-\omega) \left(\nabla_j \left[\underbrace{\epsilon_{ji}(\omega)\, E_i(\omega)}_{\vec{D}_j(\omega)} + P_j(\omega) \right] - \rho \right)$$

$$\Rightarrow \quad \vec{\nabla} \cdot \left[\vec{D}(\vec{r},\omega) + \vec{P}(\vec{r},\omega) \right] = \rho(\vec{r},\omega).$$

(6) Show that $\epsilon_{ij}(\omega) = \epsilon_{ji}(-\omega)$, that is, $\overleftrightarrow{\epsilon}(\omega) = \overleftrightarrow{\epsilon}(\omega)^{\dagger}$,

$\overleftrightarrow{\epsilon}$ is Hermitian.

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \vec{E}(-\omega)\cdot\vec{D}(\omega) = \int \frac{d\omega}{2\pi}\, \vec{E}(\omega)\cdot\vec{D}(-\omega)$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, E_i(-\omega)\,\epsilon_{ij}(\omega)\,E_j(\omega) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, E_j(\omega)\,\epsilon_{ji}(-\omega)\,E_i(-\omega)$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, E_i(-\omega)\left[\epsilon_{ij}(\omega) - \epsilon_{ji}(-\omega)\right] E_j(\omega) = 0$$

$$\epsilon_{ij}(\omega) = \epsilon_{ji}(-\omega)$$

$$\Rightarrow \overleftrightarrow{\epsilon}(\omega) = \overleftrightarrow{\epsilon}(-\omega)^{T} = \overleftrightarrow{\epsilon}(\omega)^{\dagger}$$

(7) $\delta\vec{A}$: $$0 = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\left[P_k(-\omega)\, i\omega\,\delta A_k(\omega) + E_j(-\omega)\,\epsilon_{jk}(\omega)\, i\omega A_k(\omega) - H_i(-\omega)\,\epsilon_{ijk}\,\nabla_j\,\delta A_k(\omega) + j\,\delta\vec{A}\right]$$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\delta A_k(\omega)\left[i\omega\, P_k(-\omega) + i\omega\, D_k(-\omega) + \epsilon_{ijk}\,\nabla_j\, H_i(-\omega) + \vec{j}\right]$$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\delta A_k(\omega)\left[i\omega\,\vec{P}(-\omega) + i\omega\,\vec{D}(-\omega) - \vec{\nabla}\times\vec{H}(-\omega) + \vec{j}\right]$$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\delta A_k(\omega)$$

$\omega \to -\omega$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\,\delta\vec{A}(-\omega)\cdot\left[-i\omega\,\vec{D}(\omega) - i\omega\,\vec{P}(\omega) - \vec{\nabla}\times\vec{H}(\omega) + \vec{j}\right]$$

$$\Rightarrow \vec{\nabla}\times\vec{H}(\omega) = -i\omega\left[\vec{D}(\omega) + \vec{P}(\omega)\right] + \vec{j}(\vec{r},\omega)$$

8 For $\rho = 0$ and $\vec{j} = 0$

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

$$\vec{\nabla} \times \vec{H} = -i\omega (\vec{D} + \vec{P})$$

9 let $\overleftrightarrow{\mu} = \mu_0$

$$\vec{\nabla} \times (\vec{\nabla} \times \epsilon_0 \vec{E}) = i\omega\mu_0\epsilon_0 \vec{\nabla} \times \vec{H}$$

$$= i\omega\mu_0^{\epsilon_0}(-i\omega)\left[\overleftrightarrow{\epsilon} \cdot \vec{E} + \vec{P}\right]$$

$$\vec{\nabla} \times (\vec{\nabla} \times \epsilon_0 \vec{E}) = \frac{\omega^2}{c^2}\left[\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} \cdot \epsilon_0 \vec{E} + \vec{P}\right]$$

$$= \frac{\omega^2}{c^2}\left[\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{1}\right)\epsilon_0 \vec{E} + \epsilon_0 \vec{E} + \vec{P}\right]$$

$$\frac{c^2}{\omega^2} \vec{\nabla} \times (\vec{\nabla} \times \epsilon_0 \vec{E}) = \overleftrightarrow{\chi} \cdot \epsilon_0 \vec{E} + \overleftrightarrow{1} \cdot \epsilon_0 \vec{E} + \vec{P}$$

$$\left[\frac{c^2}{\omega^2} \vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{1}) - \overleftrightarrow{1} - \overleftrightarrow{\chi}\right] \cdot \epsilon_0 \vec{E}(\vec{r}, \omega) = \vec{P}(\vec{r}, \omega)$$

(10) for $\overleftrightarrow{\mu} \neq \mu_0$

$$\vec{\nabla} \times \vec{E} = i\omega\, \overleftrightarrow{\mu} \cdot \vec{H}$$

$$\left(\frac{\overleftrightarrow{\mu}}{\mu_0}\right)^{-1} \cdot \left(\vec{\nabla} \times \epsilon_0 \vec{E}\right) = i\omega \mu_0 \epsilon_0 \vec{H}$$

$$\vec{\nabla} \times \left(\frac{\overleftrightarrow{\mu}}{\mu_0}\right)^{-1} \cdot \left(\vec{\nabla} \times \epsilon_0 \vec{E}\right) = \frac{i\omega}{c^2}(-i\omega)\left[\frac{\overleftrightarrow{\epsilon}}{\epsilon_0}\,\epsilon_0 \vec{E} + \vec{P}\right]$$

$$\left[\frac{c^2}{\omega^2}\,\vec{\nabla} \times \left(\frac{\overleftrightarrow{\mu}}{\mu_0}\right)^{-1} \cdot \left(\vec{\nabla} \times \overleftrightarrow{1}\right) - \overleftrightarrow{1} - \overleftrightarrow{\chi}\right] \cdot \epsilon_0 \vec{E}(\vec{r},\omega) = \vec{P}(\vec{r},\omega)$$

(11) Green's dyadic

$$\left[\frac{c^2}{\omega^2}\,\vec{\nabla} \times \left(\frac{\overleftrightarrow{\mu}}{\mu_0}\right)^{-1} \cdot \left(\vec{\nabla} \times \overleftrightarrow{1}\right) - \overleftrightarrow{1} - \overleftrightarrow{\chi}\right] \cdot \overleftrightarrow{\Gamma}(\vec{r},\vec{r}\,',\omega) = \overleftrightarrow{1}\,\delta^{(3)}(\vec{r}-\vec{r}\,')$$

(12) Thus, we have.

$$\epsilon_0 \vec{E}(\vec{r},\omega) = \int d^3r'\, \overleftrightarrow{\Gamma}(\vec{r},\vec{r}\,',\omega) \cdot \vec{P}(\vec{r}\,',\omega)$$

(13) Let $\overleftrightarrow{\mu} = \mu_0$ and $\overleftrightarrow{\epsilon} = \epsilon_0$ $\quad (\Rightarrow \overleftrightarrow{\chi} = 0)$ define $\overleftrightarrow{\Gamma}_0$

$$\left[\frac{c^2}{\omega^2} \vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{1}) - \overleftrightarrow{1} \right] \cdot \overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}', \omega) = \overleftrightarrow{1}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

$$\left[\vec{\nabla}\vec{\nabla} - \overleftrightarrow{1} \nabla^2 - \frac{\omega^2}{c^2} \overleftrightarrow{1} \right] \cdot \overleftrightarrow{\Gamma}_0 = \frac{\omega^2}{c^2} \overleftrightarrow{1}$$

$$\left[\vec{\nabla}\vec{\nabla} - \overleftrightarrow{1} \left(\nabla^2 + \frac{\omega^2}{c^2} \right) \right] \cdot \overleftrightarrow{\Gamma}_0 = \frac{\omega^2}{c^2} \overleftrightarrow{1}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

(14) apply $\vec{\nabla}$

$$\left[\nabla^2 - \left(\nabla^2 + \frac{\omega^2}{c^2} \right) \right] \vec{\nabla} \cdot \overleftrightarrow{\Gamma}_0 = \frac{\omega^2}{c^2} \vec{\nabla}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

$$\vec{\nabla} \cdot \overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}', \omega) = - \vec{\nabla}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

(15) using (14) in (13)

$$- \vec{\nabla}\vec{\nabla}\, \delta^{(3)}(\vec{r} - \vec{r}') - \left(\nabla^2 + \frac{\omega^2}{c^2} \right) \overleftrightarrow{\Gamma}_0 = \frac{\omega^2}{c^2} \overleftrightarrow{1}\, \delta^{(3)}$$

$$- \left(\nabla^2 + \frac{\omega^2}{c^2} \right) \overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}', \omega) = \left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2} \right) \overleftrightarrow{1}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

(16) Define:

$$- \left(\nabla^2 + \frac{\omega^2}{c^2} \right) G_0 (\vec{r}, \vec{r}', \omega) = \delta^{(3)}(\vec{r} - \vec{r}')$$

$$\overleftrightarrow{\Gamma}_0 (\vec{r}, \vec{r}', \omega) = \int d^3 r''\, G_0 (\vec{r}, \vec{r}'', \omega) \left(\vec{\nabla}''\vec{\nabla}'' + \frac{\omega^2}{c^2} \overleftrightarrow{1} \right) \delta^{(3)}(\vec{r}'', \vec{r}')$$

$$= \left(\vec{\nabla}\vec{\nabla} + \frac{\omega^2}{c^2} \overleftrightarrow{1} \right) G_0 (\vec{r} - \vec{r}', \omega)$$

(17) $$-\left(\nabla^2+\frac{\omega^2}{c^2}\right)G_0(\vec{r}-\vec{r}',\omega) = \delta^{(3)}(\vec{r}-\vec{r}')$$

$$G_0(\vec{r}-\vec{r}',\omega) = \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}$$

(18) Let $\vec{r}'=0$ for simplicity

$$\vec{\nabla} G_0 = \vec{\nabla}\frac{e^{i\frac{\omega}{c}r}}{4\pi r}$$

$$= i\frac{\omega}{c}(\vec{\nabla} r)\frac{e^{i\frac{\omega}{c}r}}{4\pi r} - \frac{e^{i\frac{\omega}{c}r}}{4\pi r^2}\nabla r$$

$$= i\frac{\omega}{c}\hat{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r} - \hat{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^2}$$

$$= i\frac{\omega}{c}\vec{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^2} - \vec{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^3}$$

(19) $$\vec{\nabla}\vec{\nabla} G_0 = i\frac{\omega}{c}\overleftrightarrow{1}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^2} + \left(i\frac{\omega}{c}\right)^2\hat{r}\vec{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^2} - 2i\frac{\omega}{c}\vec{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^3}\hat{r}$$
$$-\overleftrightarrow{1}\frac{e^{i\omega/c\, r}}{4\pi r^3} - i\frac{\omega}{c}\vec{r}\hat{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^3} + 3\vec{r}\frac{e^{i\frac{\omega}{c}r}}{4\pi r^4}\hat{r}$$

$$\vec{\nabla}\vec{\nabla} G_0 = \frac{e^{i\frac{\omega}{c}r}}{4\pi r^3}\left[-\overleftrightarrow{1}\left\{1-i\frac{\omega}{c}r\right\} + \hat{r}\hat{r}\left\{\left(i\frac{\omega}{c}r\right)^2 - 2i\frac{\omega}{c}r - i\frac{\omega}{c}r + 3\right\}\right]$$

$$\left(\vec{\nabla}\vec{\nabla}+\frac{\omega^2}{c^2}\overleftrightarrow{1}\right)G_0 = \frac{e^{i\frac{\omega}{c}r}}{4\pi r^3}\left[-\overleftrightarrow{1}\left\{1-\left(i\frac{\omega}{c}r\right)+\left(i\frac{\omega}{c}r\right)^2\right\} + \hat{r}\hat{r}\left\{3-3\left(i\frac{\omega}{c}r\right)+\left(i\frac{\omega}{c}r\right)^2\right\}\right]$$

(20) $$\overleftrightarrow{\Gamma}_0(\vec{r},\omega) = \frac{e^{i\frac{\omega}{c}r}}{4\pi r^3}\left[-\overleftrightarrow{1}\,u\left(i\frac{\omega}{c}r\right) + \hat{r}\hat{r}\,v\left(i\frac{\omega}{c}r\right)\right]$$

$$u(x) = 1-x+x^2$$
$$v(x) = 3-3x+x^2$$