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1. Maxwell's equations

$$\vec{\nabla} \cdot \vec{D}(\vec{r},\omega) = \rho(\vec{r},\omega)$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r},\omega) = 0$$

$$\vec{\nabla} \times \vec{E}(\vec{r},\omega) = i\omega \, \vec{B}(\vec{r},\omega)$$

$$\vec{\nabla} \times \vec{H}(\vec{r},\omega) = -i\omega\left[\vec{D}(\vec{r},\omega) + \vec{P}(\vec{r},\omega)\right] + \vec{j}(\vec{r},\omega) \qquad \frac{\partial}{\partial t} \to -i\omega$$

$$\vec{D}(\vec{r},\omega) = \overleftrightarrow{\epsilon}(\vec{r},\omega) \cdot \vec{E}(\vec{r},\omega)$$

$$\vec{B}(\vec{r},\omega) = \overleftrightarrow{\mu}(\vec{r},\omega) \cdot \vec{H}(\vec{r},\omega)$$

2.

$$\delta Z = \frac{i}{\hbar} \langle 0 | \, \delta W \, | 0 \rangle \qquad W = W[\vec{E}, \vec{B}, \phi, \vec{A}]$$

$$W = \int d^3r \int dt \; \mathcal{L}$$

(3) $$\frac{\text{Energy}}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2 \rightarrow \frac{1}{2}\vec{E}^* \cdot \vec{D} = \frac{1}{2}\vec{E}\cdot\vec{D}^*$$

$$\boxed{\int d^3r \int dt\ \frac{1}{2}\vec{E}\cdot\vec{D}} = \int d^3r \int_{-\infty}^{+\infty} dt\ \frac{1}{2}\vec{E}(\vec{r},t)^* \cdot \int_{-\infty}^{+\infty} dt'\ \overleftrightarrow{\epsilon}(\vec{r},t-t')\cdot\vec{E}(\vec{r},t')$$

$$\vec{E}(\omega)^* = \vec{E}(-\omega)$$

$$\vec{E}(\vec{r},t)^* = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, e^{+i\omega t}\,\vec{E}(\vec{r},\omega)^*$$

$$= \int d^3r \int_{-\infty}^{+\infty} dt\ \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega_1}{2\pi}\, e^{+i\omega_1 t}\,\vec{E}(\omega_1)^* \cdot \int_{-\infty}^{+\infty} dt' \int_{-\infty}^{+\infty}\frac{d\omega_2}{2\pi}\, e^{-i\omega_2(t-t')}\,\overleftrightarrow{\epsilon}(\omega_2)\cdot\int_{-\infty}^{+\infty}\frac{d\omega_3}{2\pi}\, e^{-i\omega_3 t'}\,\vec{E}(\omega_3)$$

$$= \int d^3r \int_{-\infty}^{+\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega_2}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega_3}{2\pi}\,\vec{E}(-\omega_1)\cdot\overleftrightarrow{\epsilon}(\omega_2)\cdot\vec{E}(\omega_3)\,\underbrace{\int_{-\infty}^{+\infty} dt\, e^{it(\omega_1-\omega_2)}}_{2\pi\,\delta(\omega_1-\omega_2)}\,\underbrace{\int_{-\infty}^{+\infty} dt'\, e^{it'(\omega_2-\omega_3)}}_{2\pi\,\delta(\omega_2-\omega_3)}$$

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$$= \int d^3r \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \vec{E}(-\omega)\cdot\overleftrightarrow{\epsilon}(\omega)\cdot\vec{E}(\omega)$$

$$\hookrightarrow \vec{E}(\vec{r},-\omega)\cdot\vec{D}(\vec{r},\omega)$$

(4) $\frac{1}{2}\vec{E}\cdot\vec{D}^{*} \rightarrow \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\vec{E}(\omega)\cdot\vec{D}(-\omega)$

$\omega \rightarrow -\omega \qquad d\omega \rightarrow -d\omega$

$z^{*}z$

$= \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\vec{E}(-\omega)\cdot\vec{D}(\omega)$

$\vec{E}\cdot\overleftrightarrow{\epsilon}\cdot\vec{E}$

$E \rightarrow (E_1 \; E_2 \; E_3)\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix}$

$\vec{A}\cdot\vec{B}$

(5) $$\frac{1}{2}\vec{E}\cdot\overleftrightarrow{\epsilon}\cdot\vec{E} = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\vec{E}_i(-\omega)\cdot\overleftrightarrow{\epsilon}_{ij}(\omega)\cdot\vec{E}_j(\omega)$$

$$\omega \to -\omega$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\vec{E}_j(\omega)\cdot\overleftrightarrow{\epsilon}_{ji}(-\omega)\cdot\vec{E}_i(-\omega)$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,E_i(-\omega)\,\epsilon_{ji}(-\omega)\,E_j(\omega)$$

(6) $$\frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,E_i(-\omega)\left[\epsilon_{ij}(\omega) - \epsilon_{ji}(-\omega)\right]E_j(\omega) = 0$$

$$\epsilon_{ij}(\omega) = \epsilon_{ji}(-\omega) = \epsilon_{ji}(\omega)^{*}.$$

$$\overleftrightarrow{\epsilon}(\omega) = \overleftrightarrow{\epsilon}(\omega)^{\dagger}$$

$$\frac{\partial}{\partial t} \to -i\omega$$

$$\frac{\partial\vec{A}}{\partial t} = -i\omega\vec{A}$$

$\vec{E}$
$\vec{B}$
ϕ
$\vec{A}$

(7) $$\mathcal{L} = -\frac{1}{2}\vec{E}^{*}\cdot\overleftrightarrow{\epsilon}\cdot\vec{E} + \vec{E}^{*}\cdot\overleftrightarrow{\epsilon}\cdot\left[-\vec{\nabla}\phi + i\omega\vec{A}\right]$$
$$+\frac{1}{2}\vec{H}^{*}\cdot\overleftrightarrow{\mu}\cdot\vec{H} - \vec{H}^{*}\cdot\vec{\nabla}\times\vec{A}$$
$$+\vec{P}^{*}\cdot\left[-\vec{\nabla}\phi + i\omega\vec{A}\right] - \left(\rho^{*}\phi - \vec{j}^{*}\cdot\vec{A}\right)$$

$-\vec{d}\cdot\vec{E}$

$\langle\vec{E}\,\vec{E}\rangle \neq \frac{\hbar}{i}\dot{\overleftrightarrow{\Gamma}}$

case (1) $\langle\vec{E}\rangle = 0$

$\vec{P}$ $\quad \vec{P}\big|_{eff} = \frac{\hbar}{i}\delta\overleftrightarrow{\epsilon}$ $\leftarrow$

$\langle\!\langle\vec{P}\rangle\!\rangle = 0$

case (2)

(8) $\delta\vec{E}$: $$\delta W = \int d^3r \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\delta\vec{E}^{*}\cdot\overleftrightarrow{\epsilon}\cdot\left[-\vec{E} - \vec{\nabla}\phi + i\omega\vec{A}\right]$$

$$\delta W = 0 \implies \vec{E} = -\vec{\nabla}\phi + i\omega\vec{A}$$
$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$\delta\vec{B}$:

⑥

⑨ $\delta\phi$: $$\delta W = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[-\vec{E}_i^{*} \overleftrightarrow{\epsilon}_{ij} \vec{\nabla}_j \delta\phi - \vec{P}^{*} \cdot \vec{\nabla} \delta\phi - \rho^{*} \delta\phi \right]$$

$$= \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\nabla_j \underbrace{\left(\epsilon_{ij} E_i^{*} \right)}_{\left(\epsilon_{ij}^{*} E_i\right)^{*}} + \vec{\nabla} \cdot \vec{P}^{*} - \rho^{*} \right] \delta\phi$$

$$\left(\epsilon_{ji} E_i\right)^{*} = D_j^{*}$$

$$\delta W = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\vec{\nabla} \cdot \left(\vec{D}^{*} + \vec{P}^{*} \right) - \rho^{*} \right] \delta\phi$$

$$\delta W = 0 \Rightarrow \vec{\nabla} \cdot \left(\vec{D} + \vec{P} \right) = \rho$$

(10) $\delta\vec{A}$: $\delta W = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\vec{E}_i^{*} \cdot \overleftrightarrow{\epsilon}_{ij} \, i\omega \, \delta\vec{A} - \vec{H}^{*} \cdot \vec{\nabla} \times \delta\vec{A} + \vec{P}^{*} \cdot i\omega \, \delta\vec{A} + \vec{j}^{*} \cdot \delta\vec{A} \right]$

$$- \int H_k^{*} \, \epsilon_{kij} \, \nabla_i \, \delta A_j = + \int \left(\nabla_i \, \epsilon_{kij} \, H_k^{*} \right) \delta A_j$$

$\frac{\partial}{\partial t} \to -i\omega$

$$= - \int \left(\epsilon_{jik} \, \nabla_i H_k^{*} \right) \delta A_j$$

$$= - \int \left(\vec{\nabla} \times \vec{H}^{*} \right) \cdot \delta\vec{A}$$

$\vec{E}_j^{*} \cdot \epsilon_{ji}$

$= \left(E_j \, \epsilon_{ji}^{*} \right)^{*}$

$= \left(E_j \, \epsilon_{ij} \right)^{*} = D_i^{*}$

$$\delta W = \int d^3r \int \frac{d\omega}{2\pi} \left[\vec{E}^{*} \cdot \overleftrightarrow{\epsilon} \, i\omega - \vec{\nabla} \times \vec{H}^{*} + \vec{P}^{*} i\omega + \vec{j}^{*} \right] \cdot \delta A.$$

$\delta W = 0$

$$\vec{E}^{*} \cdot \overleftrightarrow{\epsilon} \, i\omega - \vec{\nabla} \times \vec{H}^{*} + \vec{j}^{*} = 0$$

$$\left(\vec{D}^{*} + \vec{P}^{*} \right) i\omega - \vec{\nabla} \times \vec{H}^{*} + \vec{j}^{*}$$

$$\vec{\nabla} \times \vec{H}^{\cancel{*}} = -i\omega \left(\vec{D} + \vec{P} \right) + \vec{j}$$

$$\langle \vec{P}\,\vec{P} \rangle = \frac{\hbar}{i}\,\epsilon$$