Date: 2024 Mar 2

(1)
$$\mathcal{L} = -\frac{1}{2}\vec{E}^{*}\cdot\overleftrightarrow{\epsilon}\cdot\vec{E} + \vec{E}^{*}\cdot\overleftrightarrow{\epsilon}\cdot\left[-\vec{\nabla}\phi + i\omega\vec{A}\right] + \frac{1}{2}\vec{H}^{*}\cdot\overleftrightarrow{\mu}\cdot\vec{H} - \vec{H}^{*}\cdot(\vec{\nabla}\times\vec{A}) + \vec{P}^{*}\cdot\left[-\vec{\nabla}\phi + i\omega\vec{A}\right] - (\cancel{\rho}\,\phi - \cancel{\vec{J}}\cdot\vec{A})$$

(2)
$\vec{E}(\vec{r},\omega)$, $\vec{H}(\vec{r},\omega)$, $\vec{A}(\vec{r},\omega)$ } dynamical field

$\phi(\vec{r},\omega)$ } background potential

$\overleftrightarrow{\epsilon}(\vec{r},\omega)$, $\overleftrightarrow{\mu}(\vec{r},\omega)$, $\vec{P}(\vec{r},\omega)$, $\vec{M}\cancel{(\vec{r},\omega)}$ } source function

→ $\overleftrightarrow{M}$ can be rotated away (dual symmetry)
(refer Milton et al. in ~ 2008)

(3)
$$W = \int d^3r \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\mathcal{L} \quad \text{— Action}$$

$$Z = \langle 0_+ | 0_- \rangle^{\vec{P}}$$
— vacuum to vacuum transition amplitude.
— numerically $Z \leftrightarrow e^{\frac{i}{\hbar}W}$. This is in the spirit that $H(\vec{r},\vec{p}) = \frac{p^2}{2m} \leftrightarrow \frac{1}{2}mv^2$

$$\delta Z = \frac{i}{\hbar}\langle 0_+ | \delta W | 0_- \rangle^{\vec{P}}$$
— Schwinger's quantum action principle.

4. Variation in the action with respect of dynamical field satisfies the stationary action principle.

$$\delta\vec{E}: \quad \vec{E} = -\vec{\nabla}\phi + i\omega\vec{A} \quad \Rightarrow \quad \vec{\nabla}\times\vec{E} = i\omega\,\vec{\nabla}\times\vec{A}$$

$$\delta\vec{H}: \quad \vec{B} = \vec{\nabla}\times\vec{A} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{B} = \overleftrightarrow{\mu}\cdot\vec{H}$$

$$\delta\phi: \quad \vec{\nabla}\cdot(\vec{D}+\vec{P}) = 0 \qquad \vec{D} = \overleftrightarrow{\epsilon}\cdot\vec{E}$$

$$\delta\vec{A}: \quad \vec{\nabla}\times\vec{H} = -i\omega(\vec{D}+\vec{P})$$

5. Thus, the Maxwell equations for studying quantum fluctuations in vacuum are:

$$\vec{\nabla}\times\vec{E} = i\omega\vec{B} \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{H} = -i\omega(\vec{D}+\vec{P}) \quad \Rightarrow \quad \vec{\nabla}\cdot(\vec{D}+\vec{P}) = 0.$$

6. Variation in the action with respect to the source is the field.

fields are responses to sources.

$$\frac{\delta W[\vec{P}]}{\delta\vec{P}(\vec{r},\omega)} = \vec{E}(\vec{r},\omega)\,\frac{1}{2\pi}$$

$$\frac{\hbar}{i}\,\frac{\delta Z[\vec{P}]}{\delta\vec{P}(\vec{r},\omega)} = \langle 0_+|\,\vec{E}(\vec{r},\omega)\,|0_-\rangle^{\vec{P}}\,\frac{1}{2\pi}$$

$$Z = e^{\frac{i}{\hbar}\iint \vec{E}\cdot\vec{P}\,\frac{1}{2}} \quad \leftrightarrow \quad \frac{1}{Z[\vec{P}]}\,\frac{\hbar}{i}\,\frac{\delta Z[\vec{P}]}{\delta\vec{P}(\vec{r},\omega)} = \frac{\langle 0_+|\,\vec{E}(\vec{r},\omega)\,|0_-\rangle^{\vec{P}}}{\langle 0_+|0_-\rangle^{\vec{P}}}\,\frac{1}{2\pi} \equiv \vec{E}[\vec{r},\omega;\vec{P}]$$

(7) Variation with respect to background potentials yield.

$$\delta W = \int d^3r \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \; \frac{1}{2} \; \vec{E}^{*} \cdot \delta \overleftrightarrow{\epsilon} \cdot \vec{E}$$

$$\frac{\delta W}{\delta \overleftrightarrow{\epsilon}} = \frac{1}{2} \; \vec{E}^{*} \; \vec{E} \; \frac{1}{2\pi}$$

$$\frac{\hbar}{i} \frac{\delta Z}{\delta \overleftrightarrow{\epsilon}(\vec{r},\omega)} = \frac{1}{2} \langle 0_+ | \; \vec{E}(\vec{r},\omega)^{*} \; \vec{E}(\vec{r},\omega) \; | 0_- \rangle \; \frac{1}{2\pi}$$

(8) Recall:

$$\frac{\hbar}{i} \frac{\delta}{\delta \vec{P}(\vec{r},\omega)} \frac{\hbar}{i} \frac{\delta}{\delta \vec{P}^{*}(\vec{r}',\omega)} Z = \langle 0_+ | \; \vec{E}(\vec{r},\omega)^{*} \; \vec{E}(\vec{r}',\omega) \; | 0_- \rangle \; \frac{1}{2\pi}$$

(9) Together, we observe that variation in the effect of the background potential can simulate the bilinear variations in sources,

$$\frac{\hbar}{i} \frac{\delta Z}{\delta \overleftrightarrow{\epsilon}(\vec{r},\omega)} = \frac{1}{2} \int d^3r \; \delta^{(3)}(\vec{r}-\vec{r}') \; \frac{\hbar}{i} \frac{\delta}{\delta \vec{P}(\vec{r},\omega)} \; \frac{\hbar}{i} \frac{\delta}{\delta \vec{P}(\vec{r}',\omega)^{*}} Z$$

(refer: Schwinger, DeRaad, Milton Ann. of Phys. (1978))

which is sometimes stated as.

$$\vec{P}(\vec{r},\omega) \; \vec{P}(\vec{r}',\omega) \Big|_{\text{eff}} = \frac{\hbar}{i} \; \delta \overleftrightarrow{\epsilon}(\vec{r},\omega) \; \delta^{(3)}(\vec{r}-\vec{r}')$$

(10). Models for variation in the background potential

(a) variation is position of plate:

$$\delta \overleftrightarrow{\epsilon}(\vec{r},\omega) = \overleftrightarrow{\epsilon}(\vec{r}+\delta\vec{r},\omega) - \overleftrightarrow{\epsilon}(\vec{r},\omega)$$

$$= (\delta\vec{r}\cdot\vec{\nabla})\,\overleftrightarrow{\epsilon}(\vec{r},\omega)$$

This leads to variation in the action

$$\delta W = \int d^3r \int \frac{d\omega}{2\pi}\, \frac{1}{2}\, \vec{E}^{*}\cdot\left\{(\delta\vec{r}\cdot\vec{\nabla})\,\overleftrightarrow{\epsilon}\right\}\cdot\vec{E}$$

which can be used to read out the potential energy, conservative force associated to a

$$\delta U = -\,\delta\vec{r}\cdot\vec{F}$$

(b) variation in frequency distribution:

Treat the Casimir energy as a functional (with respect to frequency alone.. variation in frequency distribution

$$\delta E[\epsilon] = E[\epsilon+\delta\epsilon] - E[\epsilon]$$

For arbitrary variations in the spectrum we will have.

$$\frac{\delta\,\epsilon(\vec{r},\omega)}{\delta\,\epsilon(\vec{r},\omega')} = \delta(\omega-\omega')$$

THINK!

Keep one medium fixed.

(11) Starting from the Maxwell equations

$$\vec{B} = \overleftrightarrow{\mu} \cdot \vec{H}$$

$$\vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E}$$

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

$$\vec{\nabla} \times \vec{H} = -i\omega (\vec{D} + \vec{P})$$

(12) Let $\overleftrightarrow{\mu} = \overleftrightarrow{1} \mu_0$

$$\vec{\nabla} \times \epsilon_0 \vec{E} = i\omega \epsilon_0 \mu_0 \vec{H}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \epsilon_0 \vec{E}) = \frac{i\omega}{c^2} (-i\omega) \left[\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} \cdot \epsilon_0 \vec{E} + \vec{P} \right]$$

$$= \frac{\omega^2}{c^2} \left[\left(\frac{\overleftrightarrow{\epsilon}}{\epsilon_0} - \overleftrightarrow{1} \right) \cdot \epsilon_0 \vec{E} + \epsilon_0 \vec{E} + \vec{P} \right]$$

$$= \frac{\omega^2}{c^2} \left[\overleftrightarrow{\chi} \cdot \epsilon_0 \vec{E} + \epsilon_0 \vec{E} + \vec{P} \right]$$

$$\left[\frac{c^2}{\omega^2} \vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{1}) - \overleftrightarrow{1} - \overleftrightarrow{\chi} \right] \cdot \epsilon_0 \vec{E} = \vec{P}$$

(13) For $\overleftrightarrow{\mu} \neq \overleftrightarrow{1} \mu_0$

$$\left[\frac{c^2}{\omega^2} \vec{\nabla} \times \left(\frac{\overleftrightarrow{\mu}}{\mu_0} \right)^{-1} (\vec{\nabla} \times \overleftrightarrow{1}) - \overleftrightarrow{1} - \overleftrightarrow{\chi} \right] \cdot \epsilon_0 \vec{E} = \vec{P}$$

(14) Green's dyadic is defined using

$$\left[\frac{c^2}{\omega^2}\vec{\nabla}\times\left(\frac{\overleftrightarrow{\mu}}{\mu_0}\right)^{-1}\left(\vec{\nabla}\times\overleftrightarrow{1}\right) - \overleftrightarrow{1} - \overleftrightarrow{\chi}\right]\cdot\overleftrightarrow{\Gamma}(\vec{r},\vec{r}',\omega) = \overleftrightarrow{1}\,\delta^{(3)}(\vec{r}-\vec{r}')$$

(15) Free Green's dyadic

$$\left[\frac{c^2}{\omega^2}\vec{\nabla}\times\left(\vec{\nabla}\times\overleftrightarrow{1}\right) - \overleftrightarrow{1}\right]\cdot\overleftrightarrow{\Gamma}_0(\vec{r},\vec{r}',\omega) = \overleftrightarrow{1}\,\delta^{(3)}(\vec{r}-\vec{r}')$$

(16) Thus, we have.

$$\epsilon_0\vec{E}(\vec{r},\omega) = \int d^3r'\,\overleftrightarrow{\Gamma}(\vec{r},\vec{r}',\omega)\cdot\vec{P}(\vec{r}',\omega)$$

The quantum effects are implemented in the formalism by inserting

$$\epsilon_0\vec{E}(\vec{r},\omega) \rightarrow \frac{1}{Z}\,\frac{\hbar}{i}\,\frac{\delta Z[\vec{P}]}{\delta\vec{P}(\vec{r},\omega)}$$

(17) Thus, we have.

$$\frac{\hbar}{i}\frac{\delta}{\delta\vec{P}(\vec{r},\omega)}\,\frac{1}{Z}\,\frac{\hbar}{i}\frac{\delta}{\delta\vec{P}(\vec{r}',\omega)^*}\,Z = \frac{\hbar}{i}\,\overleftrightarrow{\Gamma}(\vec{r},\vec{r}',\omega)\,\frac{1}{\epsilon_0}$$

and for $\langle\vec{E}\rangle = 0$ we have

$$\frac{1}{Z}\,\frac{\hbar}{i}\frac{\delta}{\delta\vec{P}(\vec{r},\omega)}\,\frac{\hbar}{i}\frac{\delta}{\delta\vec{P}(\vec{r},\omega)^*}\,Z = \frac{\hbar}{i}\,\overleftrightarrow{\Gamma}(\vec{r},\vec{r}',\omega)\,\frac{1}{\epsilon_0}$$

18 Using 9 and 17

$$\frac{1}{Z}\frac{\hbar}{i}\frac{\delta Z}{\delta \overleftrightarrow{\epsilon}(\vec{r},\omega)} = \frac{1}{2}\int d^3r\, \delta^{(3)}(\vec{r}-\vec{r}')\, \frac{1}{Z}\frac{\hbar}{i}\frac{\delta}{\delta \vec{P}(\vec{r},\omega)}\, \frac{\hbar}{i}\frac{\delta}{\delta \vec{P}(\vec{r}',\omega)^*} Z.$$

$$= \frac{1}{2}\int d^3r\, \delta^{(3)}(\vec{r}-\vec{r}')\, \frac{\hbar}{i}\, \overleftrightarrow{\Gamma}(\vec{r},\vec{r}',\omega)\, \frac{1}{\epsilon_0}.$$

$$= \frac{1}{2}\frac{\hbar}{i}\, \overleftrightarrow{\Gamma}(\vec{r},\vec{r},\omega)\, \frac{1}{\epsilon_0}$$

19 Writing

$$\frac{\delta W[\vec{P}]}{\delta \overleftrightarrow{\epsilon}(\vec{r},\omega)} \leftrightarrow \frac{1}{Z}\frac{\hbar}{i}\frac{\delta Z}{\delta \overleftrightarrow{\epsilon}(\vec{r},\omega)} = \frac{1}{2}\frac{\hbar}{i}\, \overleftrightarrow{\Gamma}(\vec{r},\vec{r},\omega)\, \frac{1}{\epsilon_0}$$

Thus, we have.

$$\delta W = \frac{1}{2}\frac{\hbar}{i}\int d^3r \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\, \mathrm{tr}\, \delta \overleftrightarrow{\chi}(\vec{r},\omega)\cdot \overleftrightarrow{\Gamma}(\vec{r},\vec{r},\omega)$$

20 Using Green's dyadic equation

$$[\vec{\nabla}\times\vec{\nabla} - \chi]\cdot\Gamma = 1$$

$$-\frac{\delta\overleftrightarrow{\epsilon}}{\epsilon_0}\cdot\overleftrightarrow{\Gamma} + \Gamma^{-1}\cdot\delta\overleftrightarrow{\Gamma} = 0$$

$$\mathrm{tr}\,\delta\overleftrightarrow{\chi}\cdot\overleftrightarrow{\Gamma} = \mathrm{tr}\,\overleftrightarrow{\Gamma}^{-1}\cdot\delta\overleftrightarrow{\Gamma} = \mathrm{tr}\,\delta\ln\overleftrightarrow{\Gamma}$$

(21) Thus, we have.

$$\delta W = \frac{1}{2}\frac{\hbar}{i}\int d^3r \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \mathrm{tr}\ \delta\overleftrightarrow{\chi}\cdot\overleftrightarrow{\Gamma}(\vec{r},\vec{r},\omega)$$

$$= \frac{1}{2}\frac{\hbar}{i}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\ \mathrm{tr}\ \delta\ln\overleftrightarrow{\Gamma}$$

over both space and index.