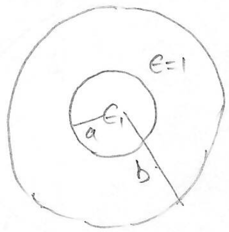
Date: 2025 May 29

① Configuration:



$$\chi_a(r) = \left(\frac{\epsilon_1}{\epsilon_0} - 1\right) \theta(a - r)$$

$$\chi_b(r) = \left(\frac{\epsilon_2}{\epsilon_0} - 1\right) \theta(r - b)$$

② Casimir energy is given by

$$E = \frac{\hbar c}{2} \int_{-\infty}^{+\infty} \frac{d\zeta/c}{2\pi} \,\mathrm{tr} \ln \left[\overleftrightarrow{1} - \overleftrightarrow{\Gamma}_a \chi_a \cdot \overleftrightarrow{\Gamma}_b \chi_b \right]$$

where

$$\left[\frac{c^2}{\omega^2} \vec{\nabla} \times \vec{\nabla} \times \overleftrightarrow{1} - \overleftrightarrow{1} - \chi_i \right] \cdot \overleftrightarrow{\Gamma}_i(\vec{r}, \vec{r}') = \overleftrightarrow{1}\, \delta^{(3)}(\vec{r} - \vec{r}')$$

③ Spherical symmetry allows us to expand in spherical harmonics.

$$Y_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}\, \Theta_{lm}(\theta)$$

$$\Theta_{lm}(\theta) = \sqrt{\frac{2l+1}{2}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin\theta)^m} \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l\, l!}$$

$$\Theta_{l0}(\theta) = \sqrt{\frac{2l+1}{2}}\, P_l(\cos\theta)$$

$0 \leq \phi < 2\pi$ $\quad l = 0, 1, 2, \ldots$

$0 \leq \theta \leq \pi$ $\quad m = -l$ to $+l$.

(4) Spherical harmonics:

(i) Completeness

$$\frac{\delta(\theta-\theta')\,\delta(\phi-\phi')}{\sin\theta} = \sum_{l=0}^{\infty}\sum_{m=-\infty}^{+\infty} Y_{lm}(\theta,\phi)\, Y_{lm}^{*}(\theta',\phi')$$

(ii) Orthogonality:

$$\int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi\; Y_{lm}^{*}(\theta,\phi)\, Y_{l'm'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}$$

(iii) Differential equation

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{lm}(\theta,\phi) = -l(l+1)\, Y_{lm}(\theta,\phi)$$

(5) Analogy between planar and spherical

$\omega \to i\varsigma.$

$\frac{\partial}{\partial x} \to ik_x$

$\frac{\partial}{\partial y} \to ik_y$

$\frac{\partial}{\partial t} \to -i\omega$

$$-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

$$= -\frac{\partial^2}{\partial z^2} + k_x^2 + k_y^2 + \frac{\varsigma^2}{c^2}$$

$$-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

$$= -\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} + \frac{\varsigma^2}{c^2}.$$

$$\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}.$$

Thus,

$$\frac{1}{r}+\frac{\partial}{\partial r} \to \frac{\partial}{\partial z}$$

$$\frac{l(l+1)}{r^2} \to k_\perp^2$$

(4) In addition, due to the dyadic nature we have to use vector spherical harmonics. We will not give the details here. Refer my notes in arXiv: 1709.08814. Using this we have.

$$E = \frac{\hbar c}{2} \int_{-\infty}^{+\infty} \frac{d\zeta/c}{2\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \operatorname{tr} \ln \left[1 - \overleftrightarrow{\gamma}_{lm,a} \chi_a \cdot \overleftrightarrow{\gamma}_{lm,b} \chi_b \right]$$

where

$$\overleftrightarrow{\gamma}_{lm}(r,r') = \begin{bmatrix} \frac{1}{\epsilon(r)}\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \frac{1}{\epsilon(r')}\left(\frac{1}{r'} + \frac{\partial}{\partial r'}\right) g_l^H & 0 & \frac{1}{\epsilon(r)}\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \frac{ik_\perp'}{\epsilon(r')} g_l^H \\ 0 & \omega^2 g_l^E & 0 \\ \frac{-ik_\perp}{\epsilon(r)} \frac{1}{\epsilon(r')} \left(\frac{1}{r'} + \frac{\partial}{\partial r'}\right) g_l^H & 0 & \frac{k_\perp k_\perp'}{\epsilon(r)\,\epsilon(r')} g_l^H \end{bmatrix}$$

$$k_\perp^2 = \frac{l(l+1)}{r^2} \qquad k_\perp'^2 = \frac{l(l+1)}{r'^2}.$$

(5) Electric and magnetic Green's function

$$\left[-\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \frac{1}{\epsilon(r)} \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} \frac{1}{\epsilon(r)} + \frac{\zeta^2}{c^2} \right] g_l^H(r,r') = \frac{\delta(r-r')}{r^2}.$$

$$\left[-\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) \left(\frac{1}{r} + \frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} + \frac{\zeta^2}{c^2} \epsilon(r) \right] g_l^E(r,r') = \frac{\delta(r-r')}{r^2}.$$

(6) We can show that E and H modes separate.

$$E = \frac{\hbar c}{2} \int_{-\infty}^{+\infty} \frac{d\zeta/c}{2\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \ln\left(1-K^E\right)\left(1-K^H\right)$$

$$K^E = (\epsilon_1 - 1)(\epsilon_2 - 1)\,\zeta^4 \int_0^a r^2 dr \int_b^\infty r'^2 dr' \; g^E_{l,a}(r',r)\; g^E_{l,b}(r,r')$$

$$K^H = \frac{(\epsilon_1 - 1)}{\epsilon_1}\frac{(\epsilon_2 - 1)}{\epsilon_2} \int_0^a r^2 dr \int_b^\infty r'^2 dr'$$

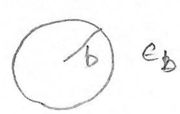
$$\times\, \mathrm{tr}\begin{bmatrix} \left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) g^H_{l,a}(r',r) & \left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right) i\sqrt{\frac{l(l+1)}{r^2}}\; g^H_{l,a}(r',r) \\ -i\sqrt{\frac{l(l+1)}{r'^2}}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) g^H_{l,a}(r',r) & \sqrt{\frac{l(l+1)}{r^2}}\sqrt{\frac{l(l+1)}{r'^2}}\; g^H_{l,a}(r',r) \end{bmatrix}$$

$$\times \begin{bmatrix} \left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right) g^H_{l,b}(r,r') & \left(\frac{1}{r}+\frac{\partial}{\partial r}\right) i\sqrt{\frac{l(l+1)}{r'^2}}\; g^H_{l,b}(r,r') \\ -i\sqrt{\frac{l(l+1)}{r^2}}\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right) g^H_{l,b}(r,r') & \sqrt{\frac{l(l+1)}{r'^2}}\sqrt{\frac{l(l+1)}{r^2}}\; g^H_{l,b}(r,r') \end{bmatrix}$$

(7) We shall evaluate K^E here. The integrations keep the r and r' disjoint

$g^E_{l,a}(r',r)$ $\qquad r < a < r'$

$g^E_{l,b}(r,r')$ $\qquad r < b < r'$

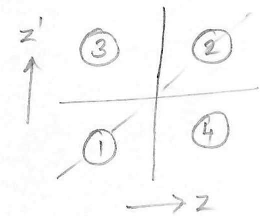


8 Planar Green function

$$\left(-\frac{\partial^2}{\partial z^2} + k_\perp^2 + \frac{\zeta^2}{c^2}\epsilon(z)\right) g^E(z,z') = \delta(z-z')$$

for $z < a < z'$

$$\left(-\frac{\partial^2}{\partial z^2} + \kappa_1^2\right) \begin{matrix} e^{-\kappa_1 z} \\ e^{+\kappa_1 z} \end{matrix} = 0$$



3 : $z<a<z'$

9 Spherical Green funct

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} + \frac{\zeta^2}{c^2}\epsilon(r)\right] g^E_{l,a}(r,r') = \frac{\delta(r-r')}{r^2}$$

$\zeta_a = \zeta\sqrt{\epsilon_a}$.

for $r < a < r'$

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} + \frac{\zeta_a^2}{c^2}\right] \begin{matrix} i_l(\zeta_a r) \\ k_l(\zeta_r r) \end{matrix} = 0$$

$$i_l(t) = \sqrt{\frac{\pi}{2t}}\, I_{l+\frac{1}{2}}(t) \xrightarrow{t\gg 1} \frac{e^t}{2t}$$

modified spherical Bessel functions

$$k_l(t) = \sqrt{\frac{\pi}{2t}}\, K_{l+\frac{1}{2}}(t) \xrightarrow{t\gg 1} \frac{\pi}{2t} e^{-t}$$

(10) for $r < a < r'$ we have.

$$g^E_{l,a}(r,r') = -\frac{1}{a}\frac{1}{(\varsigma_1 a)}\frac{(\varsigma_1 a)\; i_l(\varsigma_1 r)\; k_l(\varsigma r')}{\left[(\varsigma a)\; i_l(\varsigma_1 a)\; \bar{k}_l(\varsigma a) - (\varsigma_1 a)\; \bar{i}_l(\varsigma_1 a)\; k_l(\varsigma a)\right]}$$

$$g^E_{l,b}(r,r') = -\frac{1}{b}\frac{1}{(\varsigma_2 b)}\frac{(\varsigma_2 b)\; i_l(\varsigma r)\; k_l(\varsigma_2 r')}{\left[(\varsigma_2 b)\; i_l(\varsigma b)\; \bar{k}_l(\varsigma_2 b) - (\varsigma b)\; \bar{i}_l(\varsigma b)\; k_l(\varsigma_2 b)\right]}$$

where

$$\bar{i}_l(t) = \left(\frac{1}{t} + \frac{\partial}{\partial t}\right) i_l(t)$$

$$\bar{k}_l(t) = \left(\frac{1}{t} + \frac{\partial}{\partial t}\right) k_l(t)$$

(11) Identity:

$$(\varsigma_1^2 - \varsigma^2)\int_0^a r^2 dr\; i_l(\varsigma_1 r)\; i_l(\varsigma r) = -a^2\left[\varsigma\; i_l(\varsigma_1 a)\; \bar{i}_l(\varsigma a) - \varsigma_1\; \bar{i}_l(\varsigma_1 a)\; i_l(\varsigma a)\right]$$

$$(\varsigma_2^2 - \varsigma^2)\int_b^\infty r'^2 dr'\; k_l(\varsigma_2 r')\; k_l(\varsigma r') = b^2\left[\varsigma\; k_l(\varsigma_2 b)\; \bar{k}_l(\varsigma b) - \varsigma_2\; \bar{k}_l(\varsigma_2 b)\; k_l(\varsigma b)\right]$$

(12)
$$K^E = \underbrace{\frac{\left[\varsigma\; i_l(\varsigma_1 a)\; \bar{i}_l(\varsigma a) - \varsigma_1\; \bar{i}_l(\varsigma_1 a)\; i_l(\varsigma a)\right]}{\left[\varsigma\; i_l(\varsigma_1 a)\; \bar{k}_l(\varsigma a) - \varsigma_1\; \bar{i}_l(\varsigma_1 a)\; k_l(\varsigma a)\right]}}_{\gamma^E_{31}(a)}\; \underbrace{\frac{\left[\varsigma_2\; k_l(\varsigma b)\; \bar{k}_l(\varsigma_2 b) - \varsigma\; \bar{k}_l(\varsigma b)\; k_l(\varsigma_2 b)\right]}{\left[\varsigma_2\; i_l(\varsigma b)\; \bar{k}_l(\varsigma_2 b) - \varsigma\; \bar{i}_l(\varsigma b)\; k_l(\varsigma_2 b)\right]}}_{\gamma^E_{32}(b)}$$

$((1)3)^2$

(13) Similarly, we can show

$$K^H = r_{31}^H(a)\ r_{32}^H(b)$$

see arXiv: 1907.04301

(14) Together, we have.

$$E = \frac{\hbar c}{2} \int_{-\infty}^{+\infty} \frac{d\xi/c}{2\pi} \sum_{l=0}^{\infty} (2l+1)\ \ln\left[1 - r_{31}^E(a)\ r_{32}^E(b)\right]\left[1 - r_{31}^H(a)\ r_{32}^H(b)\right]$$