Date: 2025 May 30

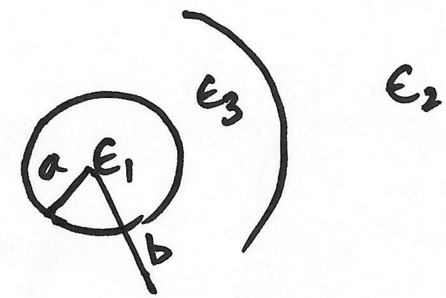
$\hbar = 1$

$c = 1$

$\epsilon_0 = 1$

$\epsilon_3 = 1$

1



$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$

$$\chi_a = (\epsilon_1 - 1)\,\theta(a - r)$$

$$\chi_b = (\epsilon_2 - 1)\,\theta(r - b)$$

2

$$E = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{d\xi}{2\pi}\, \mathrm{tr} \ln \left[\overleftrightarrow{1} - \overleftrightarrow{\Gamma}_a \chi_a \cdot \overleftrightarrow{\Gamma}_b \chi_b \right]$$

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\omega \rightarrow i\xi$$

$$\left[\overbrace{\frac{c^2}{\omega^2}}^{c=1} \vec{\nabla} \times (\vec{\nabla} \times \overleftrightarrow{1}) - \overleftrightarrow{1} - \chi_a \overleftrightarrow{1} \right] \cdot \overleftrightarrow{\Gamma}_a(\vec{r}, \vec{r}\,') = \overleftrightarrow{1}\, \delta^{(3)}(\vec{r} - \vec{r}\,')$$

$$\overleftrightarrow{\mu} = \overleftrightarrow{1}\mu_0 = \overleftrightarrow{1}$$

(3) Spherical harmonics

$$Y_{lm}(\theta,\phi) = \frac{e^{im\phi}}{\sqrt{2\pi}}\,\Theta_{lm}(\theta)$$

$$\Theta_{lm}(\theta) = \sqrt{\frac{2l+1}{2}}\sqrt{\frac{(l+m)!}{(l-m)!}}\,\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l\,l!}$$

$$\Theta_{l0}(\theta) = \sqrt{\frac{2l+1}{2}}\;P_l(\cos\theta)$$

$\downarrow$
$m=0$

(4) $$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{+l} Y_{lm}(\theta,\phi)\,f_{lm} \qquad \left\{\begin{matrix} 0\le\phi<2\pi \\ 0\le\theta\le\pi \end{matrix}\right\}$$

$$f_{lm} = \int_0^{\pi}\sin\theta\,d\theta\int_0^{2\pi}d\phi\;Y^*_{lm}(\theta,\phi)\,f(\theta,\phi) \qquad \begin{matrix} l=0,1,2,\dots \\ m=-l,\dots,+l. \end{matrix}$$

5 Vector spherical harmonics.

arXiv : 1709.08814

6 <u>Planar</u>

$$-\nabla^2 + \cancel{\frac{1}{c^2}}\frac{\partial^2}{\partial t^2} = -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial t^2}.$$

$$\frac{\partial}{\partial x} \to i k_x \qquad \frac{\partial}{\partial y} \to i k_y \qquad \frac{\partial}{\partial t} \to -i\omega$$

$$= -\frac{\partial^2}{\partial z^2} + \underbrace{k_x^2 + k_y^2 + \omega^2}_{K^2}$$

$$\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

$$= -\frac{\partial^2}{\partial z^2} + K^2$$

<u>Spherical</u>

$$-\nabla^2 + \frac{\partial^2}{\partial t^2} = -\left(\frac{1}{r} + \frac{\partial}{\partial r}\right)\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial t^2}$$

$$\boxed{\frac{1}{r} + \frac{\partial}{\partial r} \longrightarrow \frac{\partial}{\partial z}} \qquad \boxed{\frac{l(l+1)}{r^2} \to k_\perp^2}$$

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{lm}(\theta,\phi) = -l(l+1)\, Y_{lm}(\theta,\phi)$$

(7) $$E = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\underbrace{\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}}_{\theta,\phi}\ \underset{\text{dyadic index}}{\mathrm{tr}}\ \ln\left[\overleftrightarrow{I} - \overleftrightarrow{\gamma}_{lm,a}\,\chi_a \cdot \overleftrightarrow{\gamma}_{lm,b}\,\chi_b\right]$$

1-body Green dyadic.

$$\overleftrightarrow{\gamma}_{lm,a} = \begin{bmatrix} \frac{1}{\epsilon_a(r)}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\frac{1}{\epsilon_a(r')}\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g_l^H(r,r') & 0 & \frac{ik_\perp'}{\epsilon_a(r)\,\epsilon_a(r')}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)g_l^H(r,r') \\ 0 & \zeta^2 g_l^E(r,r') & 0 \\ \frac{-ik_\perp}{\epsilon_a(r')\,\epsilon_a(r)}\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g_l^H(r,r') & 0 & \frac{k_\perp k_\perp'}{\epsilon_a(r)\,\epsilon_a(r')}\,g_l^H(r,r') \end{bmatrix}$$

$$k_\perp^2 = \frac{l(l+1)}{r^2} \qquad k_\perp'^2 = \frac{l(l+1)}{r'^2}$$

careful with $l=0$ mode contribution in ~~H or~~ E mode.

8 **Planar**

$$\left[-\frac{\partial^2}{\partial z^2} + \underbrace{k_x^2 + k_y^2}_{k_\perp^2} + \zeta^2 \epsilon_a(z)\right] g^E(z,z';k_x,k_y,\zeta) = \delta(z-z')$$

x, y, z.

dr, $\boxed{r}\, d\theta$, $\boxed{r \sin\theta}\, d\phi$

Spherical

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2} + \zeta^2 \epsilon_a(r)\right] g_l^E(r,r') = \frac{\delta(r-r')}{r^2}.$$

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\frac{1}{\epsilon_a(r)}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right) + \frac{l(l+1)}{r^2}\frac{1}{\epsilon_a(r)} + \zeta^2\right] g_l^H(r,r') = \frac{\delta(r-r')}{r^2}$$

(9) $$E = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\zeta}{2\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l} \ln\left(1-K^E\right)\left(1-K^H\right)$$

$l=0 \to\ ?$

$\sum_{m=-l}^{+l} \to (2l+1)$

$$k_x^2+k_y^2 \to k_\perp^2$$

$$K^E = \left(\epsilon_1-\overset{\epsilon_3}{1}\right)\left(\epsilon_2-\overset{\epsilon_3}{1}\right)\int_0^a r^2dr\int_b^\infty r'^2dr'\ \zeta^2 g^E_{l,a}(r',r)\ \zeta^2 g^E_{l,b}(r,r')$$

$g^E_{l,a}(r',r) \to$ a interface

$g^E_{l,b}(r,r') \to$ b interface.

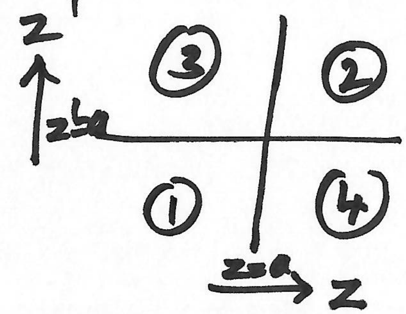
$$K^H = \frac{(\epsilon_1-1)}{\epsilon_1}\frac{(\epsilon_2-1)}{\epsilon_2}\int_0^a r^2dr\int_b^\infty r'^2dr'\ \mathrm{tr}$$

$$\times\begin{bmatrix} \left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g^H_{l,a}(r',r) & i\sqrt{\frac{l(l+1)}{r^2}}\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g^H_{l,a}(r',r) \\ -i\sqrt{\frac{l(l+1)}{r'^2}}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)g^H_{l,a}(r',r) & \sqrt{\frac{l(l+1)}{r^2}}\sqrt{\frac{l(l+1)}{r'^2}}\ g^H_{l,a}(r',r) \end{bmatrix}$$

$$\times\begin{bmatrix} \left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g^H_{l,b}(r,r') & i\sqrt{\frac{l(l+1)}{r'^2}}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)g^H_{l,b}(r,r') \\ -i\sqrt{\frac{l(l+1)}{r^2}}\left(\frac{1}{r'}+\frac{\partial}{\partial r'}\right)g^H_{l,b}(r,r') & \sqrt{\frac{l(l+1)}{r^2}}\sqrt{\frac{l(l+1)}{r'^2}}\ g^H_{l,b}(r,r') \end{bmatrix}$$

10 Planar

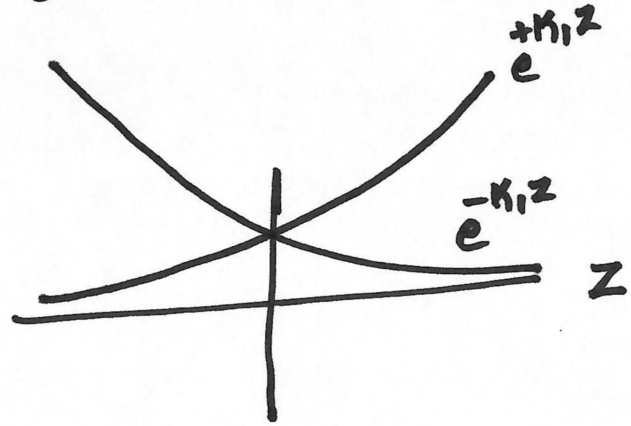
$$\left[-\frac{\partial^2}{\partial z^2} + k_\perp^2 + \zeta^2 \underset{a}{\epsilon}(z)\right] g_a^E(z,z') = \delta(z-z')$$



$z < a < z'$

$$\left(-\frac{\partial^2}{\partial z^2} + \kappa_1^2\right) \begin{matrix} e^{+\kappa_1 z} \\ e^{-\kappa_1 z} \end{matrix} = 0$$

3 : $z < a < z'$



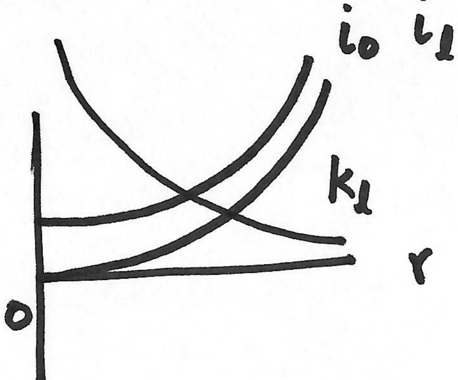
$$\kappa_1^2 = k_\perp^2 + \zeta^2 \epsilon_1$$

(11) Spherical

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}+\varsigma^2\epsilon_a(r)\right] g^E_{l,a}(r,r') = \frac{\delta(r-r')}{r^2}$$

$r < a < r'$ $\qquad \varsigma_a = \varsigma\sqrt{\epsilon_1}$

$$\left[-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)+\frac{l(l+1)}{r^2}+\varsigma_1^2\right] \begin{matrix} i_l(\varsigma_1 r) \\ k_l(\varsigma_1 r) \end{matrix} = 0$$



$$\bar{i}_l(t) = \left(\frac{1}{r}+\frac{\partial}{\partial r}\right) i_l(t)$$

$$\bar{k}_l(t) = \left(\frac{1}{r}+\frac{\partial}{\partial r}\right) k_l(t)$$

modified spherical Bessel function

NIST $\underbrace{\text{DLMF}}$

$$i_l(t) = \sqrt{\frac{\pi}{2t}}\, I_{l+\frac{1}{2}}(t) \xrightarrow{t\gg 1} \frac{e^t}{2t}$$

$$k_l(t) = \sqrt{\frac{\pi}{2t}}\, K_{l+\frac{1}{2}}(t) \xrightarrow{t\gg 1} \frac{\pi}{2t} e^{-t}$$

$$i_l(t)\,\bar{k}_l(t) - \bar{i}_l(t)\,k_l(t) = \frac{\pi}{2t^2}$$

$\varsigma_1 = \varsigma\sqrt{\epsilon_1}$ $\qquad \varsigma \to \varsigma_3 = \varsigma\sqrt{\epsilon_3}$

12 for $r < a < b < r'$

$$g^E_{l,a}(r';r) = -\frac{1}{a}\frac{1}{(\varsigma_1 a)}\frac{(\varsigma_1 a)\; i_l(\varsigma_1 r)\; k_l(\varsigma r')}{\left[(\varsigma a)\; i_l(\varsigma_1 a)\; \bar{k}_l(\varsigma a) - (\varsigma_1 a)\; \bar{i}_l(\varsigma_1 a)\; k_l(\varsigma a)\right]}$$

$$g^E_{l,b}(r,r') = -\frac{1}{b}\frac{1}{(\varsigma_2 b)}\frac{(\varsigma_2 b)\; i_l(\varsigma r)\; k_l(\varsigma_2 r')}{\left[(\varsigma_2 b)\; i_l(\varsigma b)\; \bar{k}_l(\varsigma_2 b) - (\varsigma b)\; \bar{i}_l(\varsigma b)\; k_l(\varsigma_2 b)\right]}$$

13 Identity (DLMF)

$$(\varsigma_1^2 - \varsigma^2)\int_0^a r^2 dr\; i_l(\varsigma_1 r)\; i_l(\varsigma r) = -a^2\left[\varsigma\; i_l(\varsigma_1 a)\; \bar{i}_l(\varsigma a) - \varsigma_1\; \bar{i}_l(\varsigma_1 a)\; i_l(\varsigma a)\right]$$

$$(\varsigma_2^2 - \varsigma^2)\int_b^\infty r'^2 dr'\; k_l(\varsigma_2 r')\; k_l(\varsigma r') = b^2\left[\varsigma\; k_l(\varsigma_2 b)\; \bar{k}_l(\varsigma b) - \varsigma_2\; \bar{k}_l(\varsigma_2 b)\; k_l(\varsigma b)\right]$$

14 $$E = \frac{1}{2}\int_{-\infty}^{+\infty}\frac{d\varsigma}{2\pi}\sum_{l=0/1}^{\infty}(2l+1)\ \ln\left(1-K^E\right)\left(1-K^H\right)$$

$$\varsigma_1 = \varsigma\sqrt{\epsilon_1}$$

$$\varsigma_2 = \varsigma\sqrt{\epsilon_2}$$

$$K^E = r_{31}^E(a)\quad r_{32}^E(b)$$

$$K^H = r_{31}^H(a)\quad r_{32}^H(b)$$

$$r_{31}^E(a) = \frac{\varsigma\ i_l(\varsigma_1 a)\ \bar{i}_l(\varsigma a) - \varsigma_1\ \bar{i}_l(\varsigma_1 a)\ i_l(\varsigma a)}{\varsigma\ i_l(\varsigma_1 a)\ \bar{k}_l(\varsigma a) - \varsigma_1\ \bar{i}_l(\varsigma_1 a)\ k_l(\varsigma a)}$$

$$r_{32}^E(a) = \frac{\varsigma_2\ k_l(\varsigma b)\ \bar{k}_l(\varsigma_2 b) - \varsigma\ \bar{k}_l(\varsigma b)\ k_l(\varsigma_2 b)}{\varsigma_2\ i_l(\varsigma b)\ \bar{k}_l(\varsigma_2 b) - \varsigma\ \bar{i}_l(\varsigma b)\ k_l(\varsigma_2 b)}$$

$$\frac{1}{a}\int_{-\infty}^{+\infty}\frac{d\varsigma}{2\pi} \rightarrow \sum_{n=-\infty}^{\infty} kT$$

matsubara mode.