Notes on Electrodynamics

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1. These are notes prepared for the benefit of students enrolled in Electromagnetic Theory (PHYS-520A and PHYS-520-B) at Southern Illinois University–Carbondale. It will be updated periodically, and will evolve during the semester. It is not a substitute for standard textbooks, but a supplement prepared as a study-guide.
2. The following textbooks were extensively used in this compilation.
   (a) Classical Electrodynamics,
   Julian Schwinger, Lester L. Deraad Jr., Kimball A. Milton, and Wu-yang Tsai,
   Advanced Book Program.

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Chapter 1

Mathematical preliminaries

Refer notes on Mathematical Methods.

Chapter 2

Maxwell's equations

2.1 Lorentz force

1. (**20 points.**) (Based on Schwinger et al., problem 7, chapter 1.)
A charge q moves in the vacuum under the influence of uniform fields $\mathbf{E}$ and $\mathbf{B}$. The force on this charge is given by the Lorentz force

$$\mathbf{F} = q\big[\mathbf{E} + \mathbf{v} \times \mathbf{B}\big]. \tag{2.1}$$

Assume that $\mathbf{E} \cdot \mathbf{B} = 0$ and $\mathbf{v} \cdot \mathbf{B} = 0$.

   (a) For what (magnitude and direction of) velocity does the charge move without acceleration, that is, $\mathbf{F} = 0$?

   (b) What is the speed when $\sqrt{\varepsilon_0}|\mathbf{E}| = |\mathbf{B}|/\sqrt{\mu_0}$?
   (Remember, speed of light c in Maxwell's equations is identified using $\varepsilon_0\mu_0 = 1/c^2$.)

   (c) Give a realization of the physical situation in item (1b) and comment on it intuitively. (This part of the question might not be graded.)

2. (**25 points.**) A particle of mass m and charge q moving in a uniform magnetic field $\mathbf{B}$ experiences a velocity dependent force $\mathbf{F}$ given by the expression

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \tag{2.2}$$

where $\mathbf{v}(t) = d\mathbf{x}/dt$ is the velocity of the particle in terms of its position $\mathbf{x}(t)$. Choose the magnetic field to be along the positive z direction, $\mathbf{B} = B\hat{\mathbf{z}}$.

   (a) Using initial conditions $\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + v_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$ and $\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$, solve the differential equation in Eq. (2.2) to find the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ as a function of time.

   (b) In particular, prove that the particle takes a circular path. Determine the radius of this circular path and the position of the center of the circular path.

   (c) What is the precession frequency ω_p of the particle (and thus that of the velocity vector $\mathbf{v}$) about the magnetic field $\mathbf{B}$? Is the precession frequency determined by the initial conditions to the differential equation?

2.2 Maxwell's equations: Immediate consequences

2.2.1 Conservation of charge

1. (**30 points.**) The relation between charge density and current density,

$$\frac{\partial}{\partial t}\rho(\mathbf{r}, t) + \boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) = 0, \tag{2.3}$$

is the general statement of the conservation of charge.

(a) Derive the statement of conservation of charge in Eq. (2.3) from the Maxwell equations. Hint: Take time derivative of Gauss's law and divergence of Ampere's law.

(b) For an arbitrarily moving point particle with charge, the charge and current densities are

$$\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)) \tag{2.4}$$

and

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}_a(t)\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_a(t)), \tag{2.5}$$

where $\mathbf{r}_a(t)$ is the position vector and

$$\mathbf{v}_a(t) = \frac{d\mathbf{r}_a}{dt} \tag{2.6}$$

is the velocity of the charged particle. Verify the statement of the conservation of charge in Eq. (2.3) for a point particle.

2. (**10 points.**) For a wire of negligible cross section, any volume integral involving the current density $\mathbf{j}$ becomes a line integral

$$\int d^3r\,\mathbf{j} = \int d\mathbf{l}\,I, \tag{2.7}$$

after one identifies the current density as the charge flux vector for the current

$$I = \int d\mathbf{a}\cdot\mathbf{j}. \tag{2.8}$$

Deduce the relation

$$\mathbf{j} = \rho\mathbf{v}, \tag{2.9}$$

where ρ is the charge density and $\mathbf{v} = d\mathbf{l}/dt$ is the velocity of the charge flowing in the wire.

3. (**10 points.**) (Motivated from problem 2.46 Griffiths 4th edition.) If the electric field is given (in spherical coordinates) by the expression

$$\mathbf{E}(\mathbf{r}) = -\frac{\alpha}{\varepsilon_0}\,\mathbf{r}\,\theta(R-r), \tag{2.10}$$

for constant α, show that the charge density is

$$\rho(\mathbf{r}) = -3\alpha\theta(R-r) + \alpha r\delta(r-R), \tag{2.11}$$

where $\theta(x)$ is the Heaviside step function and $\delta(x)$ is the Dirac delta function.

2.2.2 Gauge invariance

1. (**10 points.**) The electric and magnetic fields are defined in terms of the scalar and vector potentials by the relations

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}. \tag{2.12}$$

Show that the potentials are not uniquely defined in that if we let

$$\mathbf{A}\to\mathbf{A}+\boldsymbol{\nabla}\lambda, \qquad \phi\to\phi-\frac{\partial}{\partial t}\lambda, \tag{2.13}$$

the electric and magnetic fields in Eq. (2.12) remain unaltered, for an arbitrary function $\lambda = \lambda(\mathbf{r},t)$. This is called gauge invariance and the corresponding substitution in Eq. (2.13) is a gauge transformation.

2.3 Maxwell's equations in various units

1. (**50 points.**) The Maxwell equations, in SI units, are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{D} &= \rho, && (2.14a)\\ \boldsymbol{\nabla}\cdot\mathbf{B} &= 0, && (2.14b)\\ -\boldsymbol{\nabla}\times\mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} &= 0, && (2.14c)\\ \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} &= \mathbf{J}, && (2.14d) \end{aligned}$$

where

$$\begin{aligned} \mathbf{D} &= \varepsilon_0\mathbf{E} + \mathbf{P}, && (2.15a)\\ \mathbf{H} &= \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}. && (2.15b) \end{aligned}$$

The Lorentz force, in SI units, is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{2.16}$$

We have

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{2.17}$$

The above quantities will be addressed with subscripts SI in the following. The corresponding quantities in Gaussian (G) units and Heaviside-Lorentz (HL) units are obtained using the conversions

$$\begin{aligned} \sqrt{\frac{\varepsilon_0}{4\pi}}\,\mathbf{D}_\mathrm{G} &= \mathbf{D}_\mathrm{SI} = \sqrt{\varepsilon_0}\,\mathbf{D}_\mathrm{HL}, & \sqrt{4\pi\varepsilon_0}\,\rho_\mathrm{G} &= \rho_\mathrm{SI} = \sqrt{\varepsilon_0}\,\rho_\mathrm{HL}, && (2.18a)\\ \frac{1}{\sqrt{4\pi\varepsilon_0}}\,\mathbf{E}_\mathrm{G} &= \mathbf{E}_\mathrm{SI} = \frac{1}{\sqrt{\varepsilon_0}}\,\mathbf{E}_\mathrm{HL}, & \sqrt{4\pi\varepsilon_0}\,\mathbf{P}_\mathrm{G} &= \mathbf{P}_\mathrm{SI} = \sqrt{\varepsilon_0}\,\mathbf{P}_\mathrm{HL}, && (2.18b)\\ \frac{1}{\sqrt{4\pi\mu_0}}\,\mathbf{H}_\mathrm{G} &= \mathbf{H}_\mathrm{SI} = \frac{1}{\sqrt{\mu_0}}\,\mathbf{H}_\mathrm{HL}, & \sqrt{4\pi\varepsilon_0}\,\mathbf{J}_\mathrm{G} &= \mathbf{J}_\mathrm{SI} = \sqrt{\varepsilon_0}\,\mathbf{J}_\mathrm{HL}, && (2.18c)\\ \sqrt{\frac{\mu_0}{4\pi}}\,\mathbf{B}_\mathrm{G} &= \mathbf{B}_\mathrm{SI} = \sqrt{\mu_0}\,\mathbf{B}_\mathrm{HL}, & \sqrt{\frac{4\pi}{\mu_0}}\mathbf{M}_\mathrm{G} &= \mathbf{M}_\mathrm{SI} = \frac{1}{\sqrt{\mu_0}}\mathbf{M}_\mathrm{HL}. && (2.18d) \end{aligned}$$

Note that the Heaviside-Lorentz units are obtained from Gaussian units by dropping the 4π's, which is called rationalization in this context.

(a) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Gaussian units. The Maxwell equations, in Gaussian units, are

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{D}_\mathrm{G} &= 4\pi\rho_\mathrm{G}, && (2.19a)\\ \boldsymbol{\nabla}\cdot\mathbf{B}_\mathrm{G} &= 0, && (2.19b)\\ -\boldsymbol{\nabla}\times\mathbf{E}_\mathrm{G} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_\mathrm{G} &= 0, && (2.19c)\\ \boldsymbol{\nabla}\times\mathbf{H}_\mathrm{G} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}_\mathrm{G} &= \frac{4\pi}{c}\mathbf{J}_\mathrm{G}, && (2.19d) \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_\mathrm{G} &= \mathbf{E}_\mathrm{G} + 4\pi\mathbf{P}_\mathrm{G}, && (2.20a)\\ \mathbf{H}_\mathrm{G} &= \mathbf{B}_\mathrm{G} - 4\pi\mathbf{M}_\mathrm{G}. && (2.20b) \end{aligned}$$

The Lorentz force, in Gaussian units, is

$$\mathbf{F} = q_{\text{G}}\mathbf{E}_{\text{G}} + q_{\text{G}}\frac{\mathbf{v}}{c} \times \mathbf{B}_{\text{G}}. \tag{2.21}$$

Here charge q_{G} has the same conversion as charge density ρ_{G}.

(b) Starting from the Maxwell equations and Lorentz force in SI units, derive the corresponding equations in Lorentz-Heaviside units. The Maxwell equations, in Heaviside-Lorentz units, are

$$\boldsymbol{\nabla} \cdot \mathbf{D}_{\text{HL}} = \rho_{\text{HL}}, \tag{2.22a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B}_{\text{HL}} = 0, \tag{2.22b}$$
$$-\boldsymbol{\nabla} \times \mathbf{E}_{\text{HL}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}_{\text{HL}} = 0, \tag{2.22c}$$
$$\boldsymbol{\nabla} \times \mathbf{H}_{\text{HL}} - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{D}_{\text{HL}} = \frac{1}{c}\mathbf{J}_{\text{HL}}, \tag{2.22d}$$

where

$$\mathbf{D}_{\text{HL}} = \mathbf{E}_{\text{HL}} + \mathbf{P}_{\text{HL}}, \tag{2.23a}$$
$$\mathbf{H}_{\text{HL}} = \mathbf{B}_{\text{HL}} - \mathbf{M}_{\text{HL}}. \tag{2.23b}$$

The Lorentz force, in Heaviside-Lorentz units, is

$$\mathbf{F} = q_{\text{HL}}\mathbf{E}_{\text{HL}} + q_{\text{HL}}\frac{\mathbf{v}}{c} \times \mathbf{B}_{\text{HL}}. \tag{2.24}$$

Here charge q_{HL} has the same conversion as charge density ρ_{HL}.

2. (**20 points.**) What will be the SI unit of magnetic charge if it were to exist? Determine the dimension of magnetic charge.

3. (**20 points.**) The Lorentz force law in SI units is

$$\mathbf{F} = q\big[\mathbf{E} + \mathbf{v} \times \mathbf{B}\big]. \tag{2.25}$$

Write down the Lorentz force law in Lorentz-Heaviside units.

4. (**20 points.**) In Gaussian units the cyclotron frequency is

$$\omega_0 = \frac{eB}{mc}, \tag{2.26}$$

where m is the mass of electron. Write down the expression for cyclotron frequency in SI units, and in Lorentz-Heaviside units.

5. (**20 points.**) In Gaussian units the power radiated by an accelerated charged particle of charge e is given by the Larmor formula,

$$P = \frac{2}{3}\frac{e^2}{c^3}\, a^2, \tag{2.27}$$

where a is the acceleration of the charged particle. Write down the Larmor formula in SI units, and in Lorentz-Heaviside units.

6. (**30 points.**) The fine-structure constant, in Gaussian units,

$$\alpha = \frac{e^2}{\hbar c}, \tag{2.28}$$

is the parameter that characterizes the strength of the electromagnetic interaction.

(a) Write down the corresponding expression for fine-structure constant in SI units, and in Lorentz-Heaviside units.

(b) Verify that the fine-structure constant is a dimensionless quantity. Show that the numerical value of the fine-structure constant is independent of the system of units.

(c) Evaluate the numerical value for the reciprocal of the fine-structure constant, α^{-1}. (A periodic table based on quantum electrodynamics breaks down for atomic numbers greater than α^{-1}.)

2.4 Magnetic charge

1. (**60 points.**) Let us consider the static configuration of a point electric charge q_e at a fixed position $\mathbf{r}_e$ and a point magnetic charge q_m at a fixed position $\mathbf{r}_m$. Let $\mathbf{r}_e - \mathbf{r}_m = \mathbf{a}$. For convenience we could choose the magnetic charge at the origin and the electric charge on the z axis.

   (a) Using Gauss's law show that the electric field for a (static) point electric charge is given by

   $$\mathbf{E} = \frac{q_e}{4\pi\varepsilon_0}\frac{(\mathbf{r}-\mathbf{r}_e)}{|\mathbf{r}-\mathbf{r}_e|^3} = -\boldsymbol{\nabla}\phi_e, \qquad \phi_e = \frac{q_e}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}_e|}. \tag{2.29}$$

   Similarly, show that the magnetic field for a point (static) magnetic charge is

   $$\mathbf{H} = \frac{q_m}{4\pi\mu_0}\frac{(\mathbf{r}-\mathbf{r}_m)}{|\mathbf{r}-\mathbf{r}_m|^3} = -\boldsymbol{\nabla}\phi_m, \qquad \phi_m = \frac{q_m}{4\pi\mu_0}\frac{1}{|\mathbf{r}-\mathbf{r}_m|}. \tag{2.30}$$

   (b) Show that the electromagnetic momentum density

   $$\mathbf{G} = \mathbf{D}\times\mathbf{B} \tag{2.31}$$

   for this configuration is

   $$\mathbf{G} = \varepsilon_0\mu_0(\boldsymbol{\nabla}\phi_e)\times(\boldsymbol{\nabla}\phi_m). \tag{2.32}$$

   Show that $\boldsymbol{\nabla}\cdot\mathbf{G} = 0$. What is the interpretation? Thus, infer that $\mathbf{G}$ can be expressed as a curl.

   (c) Show that the angular momentum density

   $$\boldsymbol{l} = \mathbf{r}\times\mathbf{G} \tag{2.33}$$

   for this configuration is

   $$\boldsymbol{l} = (\mathbf{r}\cdot\mathbf{B})\mathbf{D} - (\mathbf{r}\cdot\mathbf{D})\mathbf{B}. \tag{2.34}$$

   (d) The angular momentum is

   $$\mathbf{L} = \int d^3r\,\boldsymbol{l}, \tag{2.35}$$

   where the integration is over all space. Show that the angular momentum for this configuration to be

   $$\mathbf{L} = \frac{q_e q_m}{2\pi}\frac{1}{4\pi}\int d^3r\,\frac{(\mathbf{r}-\mathbf{r}_e)}{|\mathbf{r}-\mathbf{r}_e|^3}\frac{1}{|\mathbf{r}-\mathbf{r}_m|}. \tag{2.36}$$

   Hint: Show that the two terms in Eq. (2.34) when integrated over all space can be expressed in the form,

   $$\int d^3r\,(\mathbf{r}\cdot\mathbf{B})\mathbf{D} = \varepsilon_0\mu_0\int d^3r\Big[3\mathbf{E} + (\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E})\Big]\phi_m, \tag{2.37a}$$

   $$\int d^3r\,(\mathbf{r}\cdot\mathbf{D})\mathbf{B} = \varepsilon_0\mu_0\int d^3r\Big[\mathbf{E} + (\mathbf{r}\cdot\boldsymbol{\nabla}\mathbf{E})\Big]\phi_m. \tag{2.37b}$$

   Subtraction of these terms leads to the expression for the angular momentum in Eq. (2.36).

(e) Evaluate the angular momentum in Eq. (2.36) to be

$$\mathbf{L} = -\frac{q_e q_m}{4\pi}\,\hat{\mathbf{a}}. \tag{2.38}$$

The implication is that the static configuration of an electric charge and a magnetic monopole will have an angular momentum. Remarkably this angular momentum is independent of the magnitude of the distance between the monopole charges.
Hint: Recall that the electric field due to charge distribution $\rho(\mathbf{r}')$ at point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}'). \tag{2.39}$$

Thus, using analogy, the integrals leading to the angular momentum in Eq. (2.36) can be performed by evaluating the electric field due to a charge density that is inversely proportional to distance. To this end use Gauss's law to show that the electric field at the point $\mathbf{r}_e$ due to a charge density

$$\rho(\mathbf{r}) = \frac{\sigma}{|\mathbf{r}-\mathbf{r}_m|} \tag{2.40}$$

is

$$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\,\hat{\mathbf{a}}, \tag{2.41}$$

where $\mathbf{a} = \mathbf{r}_e - \mathbf{r}_m$.

(f) The classical configuration under consideration is upgraded to have the features of a quantum system by imposing the Bohr quantization condition

$$L = n\hbar, \qquad n = 0, 1, 2, 3, \ldots. \tag{2.42}$$

In this manner derive the charge quantization condition of Dirac,

$$\frac{q_e q_m}{4\pi} = n\hbar. \tag{2.43}$$

2. (**40 points.**) Maxwell's equations with magnetic charge are

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_e, \tag{2.44a}$$
$$\boldsymbol{\nabla}\cdot\mathbf{B} = \rho_m, \tag{2.44b}$$
$$-\boldsymbol{\nabla}\times\mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = \mathbf{J}_m, \tag{2.44c}$$
$$\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{J}_e, \tag{2.44d}$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E}, \tag{2.45a}$$
$$\mathbf{B} = \mu_0\mathbf{H}. \tag{2.45b}$$

The Lorentz force, in SI units, on an object with electric charge q_e and magnetic charge q_m is

$$\mathbf{F} = q_e\mathbf{E} + q_e\mathbf{v}\times\mathbf{B} + q_m\mathbf{H} - q_m\mathbf{v}\times\mathbf{D}. \tag{2.46}$$

We have

$$c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}. \tag{2.47}$$

(a) Let us define

$$\mathbf{F} = \mathbf{E} + i\mathbf{H}, \qquad \rho = \rho_e + i\rho_m, \tag{2.48a}$$
$$\mathbf{G} = \mathbf{D} + i\mathbf{B}, \qquad \mathbf{J} = \mathbf{J}_e + i\mathbf{J}_m, \tag{2.48b}$$

where $i = \sqrt{-1}$. Then, show that the Maxwell equations are

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho, \tag{2.49a}$$
$$-i\boldsymbol{\nabla} \times \mathbf{F} - \frac{\partial \mathbf{G}}{\partial t} = \mathbf{J}. \tag{2.49b}$$

(b) Further, define

$$q = q_e + iq_m. \tag{2.50}$$

Show that, in terms of the complex conjugate $q^* = q_e - iq_m$,

$$q^*\mathbf{F} = (q_e\mathbf{E} + q_m\mathbf{H}) + i(q_e\mathbf{H} - q_m\mathbf{E}), \tag{2.51a}$$
$$q^*\mathbf{v} \times \mathbf{G} = (q_e\mathbf{v} \times \mathbf{D} + q_m\mathbf{v} \times \mathbf{B}) + i(q_e\mathbf{v} \times \mathbf{B} - q_m\mathbf{v} \times \mathbf{D}). \tag{2.51b}$$

Thus, write the Lorentz force in the presence of magnetic charge to be

$$\mathbf{F} = \mathrm{Re}\big[q^*\mathbf{F}\big] + \mathrm{Im}\big[q^*\mathbf{v} \times \mathbf{G}\big]. \tag{2.52}$$

(c) Consider the transformations

$$\mathbf{G} \to \mathbf{G}' = e^{-i\phi}\mathbf{G}, \qquad \rho \to \rho' = e^{-i\phi}\rho, \qquad q \to q' = e^{-i\phi}q, \tag{2.53a}$$
$$\mathbf{F} \to \mathbf{F}' = e^{-i\phi}\mathbf{F}, \qquad \mathbf{J} \to \mathbf{J}' = e^{-i\phi}\mathbf{J}. \tag{2.53b}$$

Show that the Maxwell equations do not change under these transformations if ϕ is uniform in space and time.

(d) If ϕ is not uniform show that the Maxwell equations transform into

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \rho + \rho^{\text{eff}}, \tag{2.54a}$$
$$-i\boldsymbol{\nabla} \times \mathbf{F} - \frac{\partial \mathbf{G}}{\partial t} = \mathbf{J} + \mathbf{J}^{\text{eff}}, \tag{2.54b}$$

where

$$\rho^{\text{eff}} = i(\boldsymbol{\nabla}\phi) \cdot \mathbf{G}, \tag{2.55a}$$
$$\mathbf{J}^{\text{eff}} = (\boldsymbol{\nabla}\phi) \times \mathbf{F} - i\left(\frac{\partial \phi}{\partial t}\right)\mathbf{G}. \tag{2.55b}$$

The real part of ρ^{eff} and $\mathbf{J}^{\text{eff}}$ describe the characteristic features of a topological insulator.

3. (**40 points.**) Consider the motion of a particle with electric charge q_e and mass m in the field of a stationary particle with magnetic charge q_m.

(a) Show that the magnetic field of a particle with magnetic charge q_m and no electric charge is

$$\mathbf{H} = \frac{q_m}{4\pi\mu_0}\frac{\mathbf{r}}{r^3}. \tag{2.56}$$

(b) Using the Lorentz force show that the equation of motion for an electric charge q_e (with no magnetic charge) in the presence of the magnetic field due to a magnetic charge q_m (with no electric charge) at the origin is

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{\alpha}{m}\frac{d\mathbf{r}}{dt} \times \frac{\mathbf{r}}{r^3}, \qquad \alpha = \frac{q_e q_m}{4\pi}. \tag{2.57}$$

(c) Starting from the equation of motion show that

$$\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}=0. \tag{2.58}$$

Thus, show that the kinetic energy

$$K=\frac{1}{2}m\left(\frac{d\mathbf{r}}{dt}\right)^2 \tag{2.59}$$

is a constant of motion.

(d) Show that

$$\mathbf{L}=m\mathbf{r}\times\frac{d\mathbf{r}}{dt} \tag{2.60}$$

is not a constant of motion. However, show that

$$(\mathbf{L}-\alpha\hat{\mathbf{r}}) \tag{2.61}$$

is a constant of motion.

(e) Starting from the equation of motion show that

$$\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}=0. \tag{2.62}$$

Thus, derive

$$\frac{1}{2}\frac{d^2}{dt^2}r^2=v^2, \tag{2.63}$$

where v is magnitude of velocity. Thus, the orbit is described by

$$r=\sqrt{(v^2t+c)t+b^2}. \tag{2.64}$$

Thus, conclude that this motion does not permit bound states.

(f) Show that $\mathbf{L}^2$ is a constant of motion.

(g) Show that

$$\frac{1}{2}mv^2+\frac{L^2-\alpha^2}{2mr^2} \tag{2.65}$$

is a constant of motion.

(h) Show that $-\alpha\hat{\mathbf{r}}$ and $\mathbf{L}$ constitute two perpendicular sides of a right angled triangle, with the hypotenuse given by $(\mathbf{L}-\alpha\hat{\mathbf{r}})$. That is,

$$\hat{\mathbf{r}}\cdot\mathbf{L}=0,\quad \hat{\mathbf{r}}\cdot(\mathbf{L}-\alpha\hat{\mathbf{r}})=-\alpha,\quad \mathbf{L}\cdot(\mathbf{L}-\alpha\hat{\mathbf{r}})=L^2. \tag{2.66}$$

Thus, conclude that the motion of the electric charge is confined to the surface of a cone whose axis is along $-(\mathbf{L}-\alpha\hat{\mathbf{r}})$ with cone angle θ given by

$$\cot\theta=\frac{\alpha}{L}. \tag{2.67}$$

Reference: I. R. Lapidus and J. L. Pietenpol, Classical interaction of an electric charge with a magnetic charge, Am. J. Phys. **28** (1960) 17. Also, see Y. M. Shnir, Magnetic Monopoles, Springer (2005). Original work goes back to J. J. Thompson and Poincaré.

Chapter 3

Electrostatics

3.1 Electric field

1. (**20 points.**) Earnshaw's theorem states that Poisson equation does not allow stable configurations for electric field going to zero at infinity. After the lecture on Earnshaw's theorem, in this class, I was part of a conversation that argued the following. What about a test charge placed exactly midway between two positive charges on the x-axis? I answered that the test charge will tend to slip away along the y-axis. Now, what about a test charge placed at the center of six charges, two on each of the axis. I answered that the test charge will tend to slip away in between the axes. Next, what about a test charge placed at the center of a uniformly charged spherical shell. Isn't the charge in a stable configuration now? How would you defend Earnshaw's theorem in this case?
   Hint: Remind yourself of the strength of electric field inside a uniformly charged spherical shell.

2. (**20 points.**) Electric field lines due to four positive charges of equal magnitude placed at the vertices of a square are drawn in Fig. 3.1. Using Fig. 3.1 as a guide, estimate the approximate coordinates (choosing the red dot as origin) of *all* the points where a test charge will not experience a force. Also, comment on the stability (or instability) of a test charge kept at these points.

3. (**20 points.**) Determine the capacitance of a spherical capacitor, consisting of concentric spheres of radius a and b, $b > a$, to be

$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a} - \frac{1}{b}\right)}. \tag{3.1}$$

   Take the limit $b \to \infty$ to determine the so-called self-capacitance of an isolated conducting sphere.

4. (**20 points.**) Consider an infinite chain of equidistant alternating point charges $+q$ and $-q$ on the x-axis. Calculate the electric potential at the site of a point charge due to all other charges. This is equal to the work per point charge required to assemble such a cofiguration. In terms of the distane a between neighbouring charges we can derive an expression for this energy to be

$$V = \frac{q}{4\pi\varepsilon_0}\frac{M}{a}, \tag{3.2}$$

   where M is a number defined as the Madelung constant for this hypothetical one-dimensional crystal. Determine M as an infinite sum, and evaluate the sum. (Madelung contants for three-dimensional crystals involve triple sums, which are typically a challenge to evaluate because of slow convergence.)

3.2 Gauss's law

1. (**40 points.**) Consider a uniformly charged solid sphere of radius R with total charge Q.

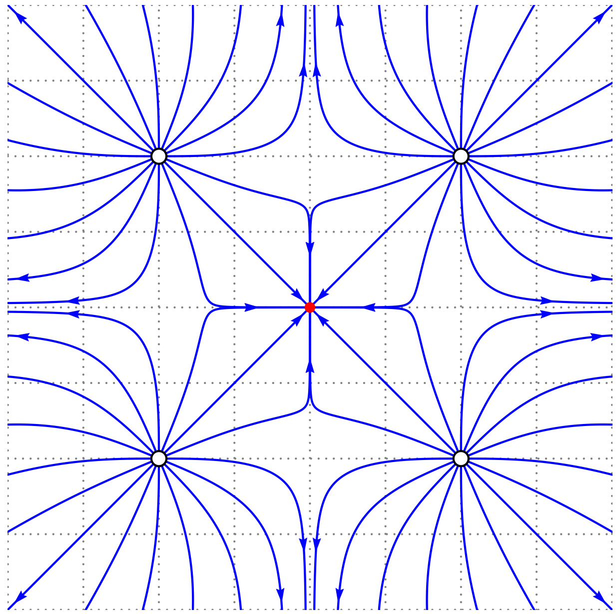

Figure 3.1: Electric field lines due to four identical positive charges placed at the vertices of a square.

(a) Using Gauss's law show that the electric field inside and outside the sphere is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{R^3}, & r < R, \\[2ex] \dfrac{Q}{4\pi\varepsilon_0}\dfrac{\mathbf{r}}{r^3}, & r > R, \end{cases} \tag{3.3}$$

where $\mathbf{r}$ is the radial vector with respect to the center of sphere.

(b) Plot the magnitude of the electric field as a function of r.

(c) Rewrite your results for the case when the solid sphere is a perfect conductor?

(d) Rewrite your results for the case of a uniformly charged hollow sphere of radius R with total charge Q.

2. (**40 points.**) Consider an infinitely long and uniformly charged solid cylinder of radius R with charge per unit length λ.

(a) Using Gauss's law show that the electric field inside and outside the cylinder is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{\mathbf{r}}{R^2}, & r < R, \\[2ex] \dfrac{\lambda}{2\pi\varepsilon_0}\dfrac{\mathbf{r}}{r^2}, & r > R, \end{cases} \tag{3.4}$$

where $\mathbf{r}$ is now the radial vector transverse to the axis of the cylinder.

(b) Plot the magnitude of the electric field as a function of r.

(c) Rewrite your results for the case when the solid cylinder is a perfect conductor?

(d) Rewrite your results for the case of a uniformly charged hollow cylinder of radius R with charge per unit length λ.

3. (**30 points.**) Consider a uniformly charged solid slab of infinite extent and thickness $2R$ with charge per unit area σ. (Note that even though the charge is spread out in the whole volume of slab we are describing it using charge per unit area σ.)

   (a) Using Gauss's law show that the electric field inside and outside the slab is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0} \dfrac{\mathbf{r}}{R}, & r < R, \\ \dfrac{\sigma}{2\varepsilon_0} \dfrac{\mathbf{r}}{r}, & r > R, \end{cases} \tag{3.5}$$

   where $\mathbf{r}$ is now the vector transverse to the plane measured from the bisecting plane of the slab.

   (b) Plot the magnitude of the electric field as a function of r.

   (c) Rewrite your results for the case when the solid slab is a perfect conductor? (Assume the same charge per unit area σ. Note that the charge is now only on the surface.)

   (d) Rewrite your results for the case of a uniformly charged hollow slab of infinite extent and thickness $2R$ with charge per unit area σ.

4. (**20 points.**) Using Gauss's law find the electric field inside and outside a uniformly charged hollow sphere of radius R with total charge Q.

5. (**20 points.**) Using Gauss's law find the electric field inside and outside a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density

$$\rho(\mathbf{r}) = br\,\theta(R - r), \tag{3.6}$$

   where r is the distance from the center of sphere. Here $\theta(x) = 1$, if $x > 0$, and 0 otherwise.

6. (**20 points.**) Consider a solid sphere of radius R with total charge Q distributed inside the sphere with a charge density

$$\rho(\mathbf{r}) = br^3\,\theta(R - r), \tag{3.7}$$

   where r is the distance from the center of sphere, and $\theta(x) = 1$, if $x > 0$, and 0 otherwise.

   (a) Integrating the charge density over all space gives you the total charge Q. Thus, determine the constant b in terms of Q and R.

   (b) Using Gauss's law find the electric field inside and outside the sphere.

   (c) Plot the electric field as a function of r.

7. (**20 points.**) Using Gauss's law find the electric field in a region, a distance R away from the origin, if the charge density in space is given

$$\rho(\mathbf{r}) = \frac{\sigma}{r}, \tag{3.8}$$

   where r is the radial distance from origin and σ is a parameter with units of charge per unit area.

8. (**20 points.**) In a homework problem we learned that the charge density

$$\rho(\mathbf{r}) = \frac{\sigma}{r}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{3.9}$$

   creates a uniform, spherically symmetric, pointing radially ourward from the origin, electric field

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{r}}, \qquad \mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}. \tag{3.10}$$

   (a) Verify this by computing $\boldsymbol{\nabla} \cdot \varepsilon_0 \mathbf{E}$ for the electric field in Eq. (3.10). Draw these electric field lines, keeping in mind that the density of electric field lines relates to the intensity of electric field.

(b) Next, determine what charge density will create a uniform, cylindrically symmetric, pointing radially ourward from the symmetry axis of cylinder, electric field

$$\mathbf{E}(\mathbf{r}) = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{\rho}}, \qquad \boldsymbol{\rho} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}. \tag{3.11}$$

Draw these electric field lines.

Caution: The Greek letter ρ is used to represent the charge density and the cylindrical coordinate.

9. (**20 points.**) (Problem 2.15 Griffiths 4th/3rd edition.)
A thick spherical shell carries charge density

$$\rho(\mathbf{r}) = \frac{k}{r^2}, \quad a \leq r \leq b. \tag{3.12}$$

Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $b < r$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

10. (**20 points.**) A thick spherical shell carries charge density

$$\rho(\mathbf{r}) = \frac{1}{4\pi r^2}\frac{Q}{(b-a)}, \quad a \leq r \leq b. \tag{3.13}$$

Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $b < r$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.

3.3 Electric potential

1. (**20 points.**) Two electrons and two protons are placed at the corners of a square of side a, such that the electrons are at diagonally opposite corners.

   (a) What is the electric potential at the center of square?

   (b) What is the electric potential at the midpoint of either one of the sides of the square?

   (c) How much potential energy is required to move another proton from infinity to the center of the square?

   (d) How much additional potential energy is required to move the proton from the center of the square to one of the midpoint of either one of the sides of the square?

2. (**20 points.**) Three protons are placed at the corners of an equilateral triangle of side a.

   (a) Determine the electric potential at the center of the triangle.

   (b) How much potential energy is required to move another proton from infinity to the center of the triangle?

3. (**20 points.**) (Griffiths 4th edition, Problem 2.32) Two positive charges, q_1 and q_2 (masses m_1 and m_2) are at rest, held together by a massless string of length a. Now the string is cut, and the particles fly off in opposite directions. How fast is each one going, when they are far apart?

4. (**20 points.**) Consider two concentric spherical (perfectly) conducting shells, of radii a and $b > a$. The inner shell has a charge $+Q$ and the outer shell has a charge $-Q$.

   (a) Determine the expression for electric field everywhere.

   (b) Plot the magnitude of electric field as a function of the distance from the center of the concentric shells.

(c) What is force experienced by another charge $+q$ a distance r from the center?

(d) Plot the electric potential as a function of distance, choosing the the potential at the center to be zero.

5. (**20 points.**) The charge density for a point charge q_a is described by

$$\rho(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a), \tag{3.14}$$

where $\mathbf{r}_a$ is the position of the charge.

(a) Evaluate the electric potential due to the point charge using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.15}$$

(Hint: Use the δ-function property to evaluate the integrals.)

(b) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (3.15),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3.16}$$

(c) Evaluate the force exerted by the charge q_a on another charge q_b, at position $\mathbf{r}_b$, using the expression for electric field you obtained using Eq. (3.16) in

$$\mathbf{F} = q_b \mathbf{E}(\mathbf{r}_b). \tag{3.17}$$

To provide a check for your calculation, the answer for the expression for the force is provided here:

$$\mathbf{F} = \frac{q_a q_b}{4\pi\varepsilon_0} \frac{\mathbf{r}_b - \mathbf{r}_a}{|\mathbf{r}_b - \mathbf{r}_a|^3}. \tag{3.18}$$

6. (**20 points.**) The charge density of a uniformly charged sphere of radius R with total charge Q is described by

$$\rho(\mathbf{r}) = \frac{Q}{\frac{4}{3}\pi R^3} \theta(R - r). \tag{3.19}$$

(a) Evaluate the integral

$$\int d^3r' \rho(\mathbf{r}') \tag{3.20}$$

over all space.

(b) Evaluate the electric potential of the sphere inside and outside the sphere using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.21}$$

(Hint: Choose the observation pont to be on the z axis, which allows the θ' and ϕ' integrals to be evaluated. Then, complete the r' integral.)

(c) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (3.21),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3.22}$$

7. (**20 points.**) The charge density for a perfectly conducting sphere of radius R with total charge Q on it is described by

$$\rho(\mathbf{r}) = \frac{Q}{4\pi R^2} \delta(r - R). \tag{3.23}$$

(a) Evaluate the integral

$$\int d^3r' \rho(\mathbf{r}') \tag{3.24}$$

over all space.

(b) Evaluate the electric potential of the sphere inside and outside the sphere using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.25}$$

(Hint: Use the δ-function property to evaluate the r' integral. Choose the observation pont to be on the z axis, which allows the θ' and ϕ' integrals to be evaluated.)

(c) Evaluate the electric field due to the point charge by finding the gradient of the electric potential you calculated using Eq. (3.25),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3.26}$$

8. (**20 points.**) The electric potential due to an infinitely thin plate (or a large disc of radius R on the xy-plane with $|x|, |y|, |z| \ll R$) with uniform charge density σ is given by the expression

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0} \big[R - |z| \big]. \tag{3.27}$$

Find the (simplified) expression for the electric field due to the plane by evaluating the gradient of the above electric potential,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3.28}$$

3.4 Point dipole

1. (**20 points.**) Consider an electric dipole, with the negative charge $-q$ at the coordinate $(0, 0, -a)$ and the positive charge $+q$ at $(0, 0, a)$, such that the electric dipole moment $\mathbf{p}$ points along the z-axis, $p = 2aq$.

(a) Write the charge density for the electric dipole in terms of δ-functions as

$$\rho(\mathbf{r}) = q\, \delta^{(3)}(\mathbf{r} - a\,\hat{\mathbf{k}}) - q\delta^{(3)}(\mathbf{r} + a\hat{\mathbf{k}}). \tag{3.29}$$

Integrate the charge density over all space using the property of δ-functions. Interpret your result.

(b) The electric potential due to a charge distribution is given using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{3.30}$$

Show that the electric potential due to the dipole at the point

$$\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}, \qquad r = \sqrt{x^2 + y^2 + z^2}, \tag{3.31}$$

is given by the expression

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-a)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+a)^2}} \right]. \tag{3.32}$$

Hint: All the integrals can be completed using the property of δ-functions.

(c) For $a \ll r$ show that the potential is approximately given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{pz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}. \tag{3.33}$$

(d) Consider the limit when a is made to vanish while q becomes infinite in such a way that $2aq$ remains the finite value p. This is a point dipole. The electric potential for a point dipole is exactly described by Eq. (3.33). Using polar coordinates write $z = r\cos\theta$ and rewrite the potential of a point dipole in Eq. (3.33) in the form

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{3.34}$$

(e) Evaluate the electric field due to a point dipole using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi \tag{3.35}$$

and express it in the following form,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\big]. \tag{3.36}$$

Draw the electric field lines of a point dipole for $\mathbf{p} = p\,\hat{\mathbf{z}}$.
Hint: Use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$ and $\boldsymbol{\nabla} r = \hat{\mathbf{r}}$.

2. (**20 points.**) The electric potential for a charge distribution is exactly described by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{c}\cdot\mathbf{r}}{r^3}, \qquad \mathbf{r} \neq 0, \tag{3.37}$$

where $\mathbf{c}$ is a property of the charge distribution.

(a) Evaluate the corresponding electric field using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{3.38}$$

(b) Draw the electric field lines. Further, draw a picture illustrating the features of the charge distribution described by $\mathbf{c}$.

3. (**20 points.**) The electric field due to a point dipole $\mathbf{d}$ at a distance $\mathbf{r}$ away from dipole is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[3(\mathbf{d}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d}\big]. \tag{3.39}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{d} = d\,\hat{\mathbf{z}}$.

(a) Qualitatively plot the electric field lines for the dipole $\mathbf{d}$. (Hint: You do not have to depend on Eq. (3.39) for this purpose. An intuitive knowledge of electric field lines should be the guide.)

(b) Find the (simplified) expression for the electric field on the positive z-axis. (Hint: On the positive z-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ and $r = z$.)

(c) Find the (simplified) expression for the electric field everywhere on the x-axis. (Hint: On the positive x-axis we have, $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ and $r = x$.) Plot the magnitude of the electric field on the x-axis as a function of x.

4. (**20 points.**) The electric field of a point dipole $\mathbf{p}$ is given by the expression

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\Big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\Big] = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r} - \mathbf{p}r^2}{r^5}\right]. \tag{3.40}$$

Evaluate $\boldsymbol{\nabla}\cdot\mathbf{E}$.
Hint: Intuitively, the divergence of a vector field is a measure of the density of source/sink of the field. For reference, the electric field lines drawn in Fig. 3.2 will be those of a point dipole in the limit of distance between the two charges going to zero, keeping the magnitude of the dipole moment fixed.

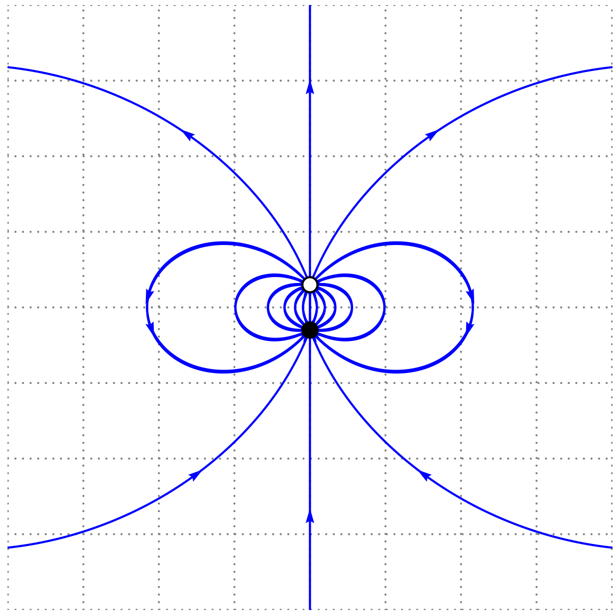

Figure 3.2: Electric field lines for a point dipole.

5. (**20 points.**) (Ref. Schwinger et al., problem 4.1.)
Consider the charge density
$$\rho(\mathbf{r}) = -\mathbf{d}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}), \tag{3.41}$$
where $\mathbf{d}$ is constant (uniform in space).

   (a) Find the total charge of the charge density by evaluating
$$\int d^3r\,\rho(\mathbf{r}). \tag{3.42}$$
   Hint: Use theorem of gradient.

   (b) Find the dipole moment of the charge density by evaluating
$$\int d^3r\,\mathbf{r}\,\rho(\mathbf{r}). \tag{3.43}$$
   Hint: Integrate by parts, and use $\boldsymbol{\nabla}\,\mathbf{r} = \mathbf{1}$.

6. (**20 points.**) A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density
$$\rho(\mathbf{r},t) = -\mathbf{p}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}_0). \tag{3.44}$$
Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r},t)$.
Hint: Force on the dipole is given by the integral of the Lorentz force density
$$\mathbf{f}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t). \tag{3.45}$$
Evaluate the integral using the properties of δ-function.

7. (**20 points.**) The force and torque on an electric dipole $\mathbf{d}$ in the presence of an electric field is given by
$$\mathbf{F} = (\mathbf{d}\cdot\boldsymbol{\nabla})\mathbf{E} \qquad \text{and} \qquad \boldsymbol{\tau} = \mathbf{d}\times\mathbf{E}, \tag{3.46}$$

respectively. Thus, describe the motion of an electric dipole when placed in between the plates of a parallel plate capacitor. Assume the plates to be perfectly conducting, of infinite cross-sectional area, and the medium in between to be vacuum.

8. (**20 points.**) Consider an infinitely thin flat sheet, of infinite extent, constructed out of a continuous distribution of point dipoles, all of them poiting in the direction of $\hat{\mathbf{z}}$, each of individual charge density

$$-\mathbf{p}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}_p), \tag{3.47}$$

where $\mathbf{r}_p$ is the position of an individual point dipole. The charge density of such a sheet is given by

$$\rho(\mathbf{r}) = -\sigma\frac{\partial}{\partial z}\delta(z), \tag{3.48}$$

where $\sigma = p\,\delta(x)\delta(y)$ is the electric dipole moment per unit area.

(a) Evaluate the electric potential for the sheet using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.49}$$

(Hint: Use the δ-function property to evaluate the z'-integral, after integrating by parts. The x' and y' integrals can be completed using standard substitutions.)

(b) Evaluate the electric field for the sheet by finding the gradient of the electric potential you calculated using Eq. (3.49),

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{3.50}$$

9. (**20 points.**) Consider an electric line-dipole at the origin, constituting of an infinitely long and infinitely thin rod with uniform positive line charge density λ (charge/length), parallel to the z axis, at $x = a$, and another such rod with negative line charge density at $x = -a$. Together these form an electric line-dipole moment $\boldsymbol{\beta} = 2a\lambda\,\hat{\mathbf{i}}$. The electric potential due to this line-dipole at the point

$$\boldsymbol{\rho} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}, \qquad \rho = \sqrt{x^2+y^2}, \tag{3.51}$$

is given by the expression

$$\phi(\boldsymbol{\rho}) = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right]. \tag{3.52}$$

(a) For $a \ll \rho$ show that the potential is approximately given by

$$\phi(\boldsymbol{\rho}) = \frac{1}{4\pi\varepsilon_0}\frac{2\beta x}{(x^2+y^2)}. \tag{3.53}$$

(b) Consider the limit when a is made to vanish while λ becomes infinite, in such a way that $2a\lambda$ remains the finite value β. This is a point line-dipole. The electric potential for a point line-dipole is exactly described by Eq. (3.53). Using cylindrical polar coordinates write $x = \rho\cos\phi$ and thus rewrite the potential of a point dipole in Eq. (3.53) in the form

$$\phi(\boldsymbol{\rho}) = \frac{1}{2\pi\varepsilon_0}\frac{\beta\cos\phi}{\rho^2} = \frac{1}{2\pi\varepsilon_0}\frac{\boldsymbol{\beta}\cdot\boldsymbol{\rho}}{\rho^2}. \tag{3.54}$$

(c) Evaluate the electric field due to a point line-dipole using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{3.55}$$

Draw the electric field lines of a point line-dipole for $\boldsymbol{\beta} = \beta\,\hat{\mathbf{i}}$. Then, draw the equipotential lines. Are the equipotential lines circular?

10. (**20 points.**) Here is problem 4.31 of Griffiths 4th edition, which is not there in the 3rd edition:

> A point charge Q is “nailed down” on a table. Around it, at radius R is a frictionless circular track on which a dipole $\mathbf{d}$ rides, constrained always to point tangent to the circle. Use Eq. (4.5) of Griffiths, 4th/3rd edition, to show that the electric force on the dipole is
>
> $$\mathbf{F} = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}}{R^3}.$$
>
> Notice that this force is always in the “forward” direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]

Footnote 21, in Griffiths 4th edition, is an acknowledgement: “This charming paradox was suggested by K. Brownstein.”
You might also refer to comments by Prof. Alan Guth, in his Fall 2014 lecture notes, at
`http://web.mit.edu/8.07/www/probsets/ps06-f14.pdf`
`http://web.mit.edu/8.07/www/probsets/sol06-f14.pdf`

(a) The electric field of a point charge Q at distance $\mathbf{R}$ from the charge is

$$\mathbf{E}(\mathbf{R}) = \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{R}}{R^3}. \tag{3.56}$$

The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

$$U = -\mathbf{d}\cdot\mathbf{E}. \tag{3.57}$$

Thus, derive the interaction energy between the charge Q and the dipole $\mathbf{d}$ to be

$$U = -\frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{d}\cdot\mathbf{R}}{R^3}. \tag{3.58}$$

(b) The variables in the problem are the coordinate ϕ that specifies the position of the dipole on the circular track, and the angle θ that the direction of the dipole makes with respect to the radius vector $\mathbf{R}$. Thus, conclude that the interaction energy is independent of the coordinate ϕ,

$$U(\theta) = -\frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{R^2}. \tag{3.59}$$

(c) The generalized tangential force on the dipole, upto a factor R, is

$$F_\phi = -\frac{\partial}{\partial\phi}U. \tag{3.60}$$

Thus, conclude that there is no tangential force acting on the dipole. No perpetual motion!

(d) The torque acting on the dipole is

$$F_\theta = -\frac{\partial}{\partial\theta}U. \tag{3.61}$$

Determine the angles for which this force in zero. Analyse each of these angles and find out if they are stable or unstable.

(e) Describe the motion of the dipole on the track for arbitrary initial conditions with respect to ϕ and θ. That is, describe your results in 10d.

3.5 Poisson equation

1. (**60 points.**) Consider a line segment of length $2L$ with uniform line charge density λ.

   (a) When the rod is placed on the z-axis centered on the origin, show that the charge density can be expressed as

   $$\rho(\mathbf{r}) = \lambda\delta(x)\delta(y)\theta(-L < z < L), \tag{3.62}$$

   where $\theta(-L < z < L) = 1$, if $-L < z < L$ and $\theta(z) = 0$, otherwise.

   (b) Inverting the Poisson equation for the electric potential, using the Green's function, evaluate the electric potential for the rod as

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right)\right]. \tag{3.63}$$

   (c) Using $\sinh t = (e^t - e^{-t})/2$, show that

   $$\sinh^{-1} t = \ln(t + \sqrt{t^2+1}). \tag{3.64}$$

   (d) Thus, express the electric potential of Eq. (3.63) in the form

   $$\phi(\mathbf{r}) = \frac{\lambda}{4\pi\varepsilon_0}\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L}, \frac{\rho}{L}\right)\right], \tag{3.65}$$

   where $\rho^2 = x^2 + y^2$ and

   $$F(a,b) = \ln[1 - a + \sqrt{(1-a)^2 + b^2}] + \ln[1 + a + \sqrt{(1+a)^2 + b^2}]. \tag{3.66}$$

   (e) An infinite rod (on the z axis) is obtained by taking the limit $\rho \ll L, z \ll L$. Show that

   $$\phi(\mathbf{r}) \xrightarrow{\rho \ll L, z \ll L} -\frac{2\lambda}{4\pi\varepsilon_0}\ln\frac{\rho}{2L}. \tag{3.67}$$

   Hint: Series expand and keep only leading order terms.

   (f) Using $\mathbf{E} = -\boldsymbol{\nabla}\phi$ determine the electric field for an infinite rod (placed on the z-axis) to be

   $$\mathbf{E}(\mathbf{r}) = \frac{2\lambda}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}. \tag{3.68}$$

2. (**20 points.**) In class we evaluated the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involved the integral

   $$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2 + r'^2 - 2rr't}}. \tag{3.69}$$

   Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2 + r'^2 - 2rr't = y$.)

Chapter 4

Multipole expansion

4.1 Dipole moment

1. (**30 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.9.)

   (a) The electric field of a point charge q at distance $\mathbf{r}$ is

   $$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{r}}{r^3}. \tag{4.1}$$

   The force on a point dipole in the presence of an electric field is

   $$\mathbf{F} = (\mathbf{d} \cdot \boldsymbol{\nabla})\mathbf{E}. \tag{4.2}$$

   Use these to find the force on a point dipole due to a point charge.

   (b) The electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by

   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \big[3\,\hat{\mathbf{r}}\,(\mathbf{d} \cdot \hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{4.3}$$

   The force on a point charge in the presence of an electric field is

   $$\mathbf{F} = q\mathbf{E}. \tag{4.4}$$

   Use these to find the force on a point charge due to a point dipole.

   (c) Confirm that above two forces are equal in magnitude and opposite in direction, as per Newton's third law.

2. (**40 points.**) (Based on Griffiths 3rd/4th ed., Problem 4.8.)
   We showed in class that the electric field of a point dipole $\mathbf{d}$ at distance $\mathbf{r}$ from the dipole is given by the expression

   $$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^3} \big[3\,\hat{\mathbf{r}}\,(\mathbf{d} \cdot \hat{\mathbf{r}}) - \mathbf{d}\big]. \tag{4.5}$$

   The interaction energy of a point dipole $\mathbf{d}$ in the presence of an electric field is given by

   $$U = -\mathbf{d} \cdot \mathbf{E}. \tag{4.6}$$

   Further, the force between the two dipoles is given by

   $$\mathbf{F} = -\boldsymbol{\nabla} U. \tag{4.7}$$

   Use these expressions to derive

(a) the interaction energy between two point dipoles separated by distance $\mathbf{r}$ to be

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\big[\mathbf{d}_1\cdot\mathbf{d}_2 - 3\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\big]. \tag{4.8}$$

(b) the force between the two dipoles to be

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{3}{r^4}\big[(\mathbf{d}_1\cdot\mathbf{d}_2)\,\hat{\mathbf{r}} + (\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + (\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 - 5\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\big]. \tag{4.9}$$

(c) Are the forces central? That is, is the force in the direction of $\mathbf{r}$?

(d) Are the forces on the dipole equal in magnitude and opposite in direction? That is, do they satisfy Newton's third law?

3. (**20 points.**) For what a, b, and $\mathbf{c}$, is the relation

$$\boldsymbol{\nabla}\left[\frac{(\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})}{r^3}\right] = \frac{a\,(\mathbf{d}_1\cdot\hat{\mathbf{r}})\,\mathbf{d}_2 + b\,(\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{d}_1 + (\mathbf{d}_1\cdot\hat{\mathbf{r}})(\mathbf{d}_2\cdot\hat{\mathbf{r}})\,\mathbf{c}}{r^4} \tag{4.10}$$

an identity. What are the dimensions of a, b, and $\mathbf{c}$?

4. (**20 points.**) The potential energy of an electric dipole $\mathbf{p}$ in an electric field, that is not necessarily uniform, is

$$U = -\mathbf{p}\cdot\mathbf{E}. \tag{4.11}$$

Restricting to electrostatics, ($\boldsymbol{\nabla}\cdot\mathbf{D} = \rho$ and $\boldsymbol{\nabla}\times\mathbf{E} = 0$,) show that the force on the electric dipole moment

$$\mathbf{F} = -\boldsymbol{\nabla}U \tag{4.12}$$

is given in terms of the directional derivative of the electric field in the direction of the electric dipole moment,

$$\mathbf{F} = (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}. \tag{4.13}$$

5. (**10 points.**) Interaction energy of a dipole $\mathbf{d}$ with an electric field $\mathbf{E}$ is

$$U = -\mathbf{d}\cdot\mathbf{E} = -dE\cos\theta. \tag{4.14}$$

The torque on the dipole due to the electric field is

$$\boldsymbol{\tau} = \mathbf{d}\times\mathbf{E}. \tag{4.15}$$

Force is a manifestation of the systems tendency to minimize its energy, and in this spirit torque is defined as,

$$\tau = -\frac{\partial}{\partial\theta}U = -dE\sin\theta. \tag{4.16}$$

Show that there is no inconsistency, in sign, between the two definitions of torque.

6. (**10 points.**) Show that the effective charge density, ρ_{eff}, and the effective current density, $\mathbf{j}_{\text{eff}}$,

$$\rho_{\text{eff}} = -\boldsymbol{\nabla}\cdot\mathbf{P}, \tag{4.17}$$

$$\mathbf{j}_{\text{eff}} = \frac{\partial}{\partial t}\mathbf{P} + \boldsymbol{\nabla}\times\mathbf{M}, \tag{4.18}$$

satisfy the equation of charge conservation

$$\frac{\partial}{\partial t}\rho_{\text{eff}} + \boldsymbol{\nabla}\cdot\mathbf{j}_{\text{eff}} = 0. \tag{4.19}$$

7. (**10 points.**) The magnetic dipole moment of charge q_a moving with velocity $\mathbf{v}_a$ is

$$\boldsymbol{\mu} = \frac{1}{2} q_a \mathbf{r}_a \times \mathbf{v}_a, \tag{4.20}$$

where $\mathbf{r}_a$ is the position of the charge. For a charge moving along a circular orbit of radius r_a, with constant speed v_a, deduce the magnetic moment

$$\boldsymbol{\mu} = IA\,\hat{\mathbf{n}}, \qquad I = \frac{q_a}{\Delta t}\frac{v_a \Delta t}{2\pi r_a} \qquad A = \pi r_a^2, \tag{4.21}$$

where $\hat{\mathbf{n}}$ points along $\mathbf{r}_a \times \mathbf{v}_a$.

8. (**30 points.**) Identify the orbital angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ in the expression for magnetic dipole moment, then generalize to total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, where $\mathbf{S}$ is the spin of the particle. Thus, deduce the relation

$$\boldsymbol{\mu} = \gamma \mathbf{J}, \tag{4.22}$$

where γ is the gyromagnetic ratio of a particle. A magnetic dipole moment feels a torque given by

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} = \boldsymbol{\mu} \times \mathbf{B}, \tag{4.23}$$

which causes the magnetic moment to precess around the magnetic field. Solve the above equations and find the precession angular frequency in terms of γ and B.

9. (**30 points.**) Consider a circular loop of wire carrying current I whose magnetic moment is given by $\boldsymbol{\mu} = IA\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ points perpendicular to the plane containing the loop (satisfying the right hand sense) and A is the area of the loop. Consider the case $\hat{\mathbf{n}} = \hat{\mathbf{x}}$. What is the magnitude and direction of the torque experienced by this loop in the presence of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{y}}$. Describe the resultant motion of the loop. (Hint: The torque experienced by a magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\mathbf{B}$ is $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$.)

4.2 Legendre polynomials

1. (**Recurrence relation.**) The Legendre polynomials $P_l(x)$ of degree l are defined, or generated, by expanding the electric (or gravitational) potential of a point charge,

$$\frac{\alpha}{|\mathbf{r} - \mathbf{r}'|} = \frac{\alpha}{r_>} \frac{1}{\sqrt{1 + \left(\frac{r_<}{r_>}\right)^2 - 2\left(\frac{r_<}{r_>}\right)\cos\gamma}} = \frac{\alpha}{r_>}\sum_{l=0}^{\infty}\left(\frac{r_<}{r_>}\right)^l P_l(\cos\gamma), \tag{4.24}$$

where

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'), \tag{4.25}$$

and

$$r_< = \text{Minimum}(r, r'), \tag{4.26a}$$
$$r_> = \text{Maximum}(r, r'). \tag{4.26b}$$

Thus, in terms of variables

$$t = \frac{r_<}{r_>}, \qquad 0 \le t < \infty, \tag{4.27}$$

and

$$x = \cos\gamma, \qquad -1 \le x < 1, \tag{4.28}$$

we can define the generating function for the Legendre polynomials as

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{4.29}$$

Setting $t = 0$ in the above relation we immediately learn that

$$P_0(x) = 1. \tag{4.30}$$

Legenendre polynomials of higher degrees can be derived by Taylor expansion of the generating function. However, for large degrees it is more efficient to derive a recurrence relation. To derive the recurrence relation for Legendre polynomials we begin by differentiating the generating function with respect to t to obtain

$$\frac{\partial g}{\partial t} = \frac{(x-t)}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{4.31}$$

Inquire why the sum on the right hand side now starts from $l = 1$. The second equality can be rewritten in the form

$$\frac{(x-t)}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt) \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x), \tag{4.32}$$

and implies

$$(x-t) \sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt) \sum_{l=1}^{\infty} l\, t^{l-1} P_l(x). \tag{4.33}$$

Express this in the form

$$t^0 \Big[x P_0(x) - P_1(x) \Big] \quad + t^1 \Big[3x P_1(x) - P_0(x) - 2P_2(x) \Big] \\ + \sum_{l=2}^{\infty} t^l \Big[(2l+1)x\, P_l(x) - l\, P_{l-1}(x) - (l+1)\, P_{l+1}(x) \Big] = 0. \tag{4.34}$$

Thus, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, we have, for t^0 and t^1,

$$P_1(x) = x P_0(x), \tag{4.35a}$$

$$2P_2(x) = 3x P_1(x) - P_0(x), \tag{4.35b}$$

and matching powers of t^l for $l \geq 2$ we obtain the recurrence relation for Legendre polynomials as

$$(l+1)\, P_{l+1}(x) = (2l+1)x\, P_l(x) - l\, P_{l-1}(x), \qquad l = 0, 1, 2, 3, \ldots. \tag{4.36}$$

Note that the recurrence relations in Eq. (4.39), for $l = 0$ and $l = 1$, reproduces Eqs. (4.35). The recurrence relations in Eq. (4.39) can be reexpressed in the form

$$l\, P_l(x) = (2l-1)x\, P_{l-1}(x) - (l-1)\, P_{l-2}(x), \qquad l = 1, 2, 3, \ldots. \tag{4.37}$$

Thus, Eq. (4.37) generates Legendre polynomials of all degrees starting from $P_0(x) = 1$, which was obtained in Eq. (4.30).

2. (**Differential equation.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{4.38}$$

(a) Starting from the generating function and differentiating with respect to t we derived the recurrence relation for Legendre polynomials in Eq. (4.39),

$$(l+1)\,P_{l+1}(x) = (2l+1)x\,P_l(x) - l\,P_{l-1}(x), \qquad l = 0,1,2,\ldots, \tag{4.39}$$

in terms of

$$P_0(x) = 1 = g(0,x). \tag{4.40}$$

Differentiating the recurrence relation with respect to x show that

$$(2l+1)\,P_l + (2l+1)x\,P_l' = l\,P_{l-1}' + (l+1)\,P_{l+1}', \qquad l = 0,1,2,\ldots, \tag{4.41}$$

where we supressed the dependence in x and prime in the superscript of $P_l'(x)$ denotes derivative with respect to the argument x.

(b) Differentiating the generating function with respect to x show that

$$\frac{\partial g}{\partial x} = \frac{t}{(1+t^2-2xt)^{\frac{3}{2}}} = \sum_{l=0}^{\infty} t^l P_l'(x). \tag{4.42}$$

Show that the second equality can be rewritten in the form

$$\frac{t}{\sqrt{1+t^2-2xt}} = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P_l'(x), \tag{4.43}$$

and implies

$$t\sum_{l=0}^{\infty} t^l P_l(x) = (1+t^2-2xt)\sum_{l=0}^{\infty} t^l P_l'(x). \tag{4.44}$$

Express this in the form

$$\begin{aligned} t^0\Big[P_0'(x)\Big] \quad &+ t^1\Big[P_1'(x) - 2xP_0'(x) - P_0(x)\Big] \\ &+ \sum_{l=2}^{\infty} t^l\Big[P_l'(x) + P_{l-2}'(x) - 2x\,P_{l-1}'(x) - P_{l-1}(x)\Big] = 0. \end{aligned} \tag{4.45}$$

Then, using the completeness property of Taylor expansion, that is, equating the coefficients of powers of t in the expansion, show that, for t^0 and t^1,

$$P_0'(x) = 0, \tag{4.46a}$$
$$P_1'(x) = P_0(x) = 1, \tag{4.46b}$$

and matching powers of t^l for $l \geq 2$ derive a recurrence relation for the derivative of Legendre polynomials as

$$2x\,P_{l-1}' + P_{l-1} = P_l' + P_{l-2}', \qquad l = 2,3,\ldots. \tag{4.47}$$

Here, we shall find it convenient to use the above recurrence relations in the form

$$2x\,P_l' + P_l = P_{l+1}' + P_{l-1}', \qquad l = 1,2,3,\ldots, \tag{4.48}$$

which is obtained by setting $l \to l+1$.

(c) Equations (4.41) and (4.48) are linear set of equations for P_{l-1}' and P_{l+1}' in terms of P_l and P_l'. Solve them to find

$$P_{l+1}' = xP_l' + (l+1)\,P_l, \qquad l = 0,1,2,\ldots, \tag{4.49a}$$
$$P_{l-1}' = x\,P_l' - l\,P_l. \qquad l = 1,2,3,\ldots. \tag{4.49b}$$

(d) Using $l \to l-1$ in Eq. (4.49a) show that

$$P_l' = x\,P_{l-1}' + l\,P_{l-1}. \tag{4.50}$$

Then, substitute Eq. (4.49b) to obtain

$$(1-x^2)\,P_l' = l\,P_{l-1} - xl\,P_l. \tag{4.51}$$

Differentiate the above equation and substitute Eq. (4.49b) again to derive the differential equation for Legendre polynomials as

$$\left[\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x} + l(l+1)\right]P_l(x) = 0. \tag{4.52}$$

Substitute $x = \cos\theta$ to rewrite the differential equation in the form

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right]P_l(\cos\theta) = 0. \tag{4.53}$$

3. (**Rodrigues formula for Legendre polynomials.**)
The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{4.54}$$

(a) Using binomial expansion show that

$$\frac{1}{\sqrt{1-y}} = \sum_{m=0}^{\infty} y^m \frac{(2m)!}{\left[m!\,2^m\right]^2} \tag{4.55}$$

and

$$(2xt-t^2)^m = \sum_{n=0}^{\infty} \frac{m!}{n!(m-n)!}(2xt)^{m-n} t^{2n}(-1)^n. \tag{4.56}$$

Thus, show that

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{m=0}^{\infty}\sum_{n=0}^{m} t^{m+n}\frac{(2m)!}{m!n!(m-n)!2^{m+n}}x^{m-n}(-1)^n. \tag{4.57}$$

(b) In Figure 4.1 we illustrate how we change the double sum in m and n to variables l and s. This is achieved using the substitutions

$$m+n = l, \tag{4.58a}$$
$$m-n = 2s, \tag{4.58b}$$

which corresponds to

$$2m = l+2s, \qquad m = \frac{l}{2}+s, \quad \text{and} \quad n = \frac{l}{2}-s. \tag{4.59}$$

The counting on the variable s, for given l, follows the pattern,

$$l \text{ even}: \quad 2s = 0,2,4,\ldots,l, \tag{4.60a}$$
$$l \text{ odd}: \quad 2s = 1,3,5,\ldots,l. \tag{4.60b}$$

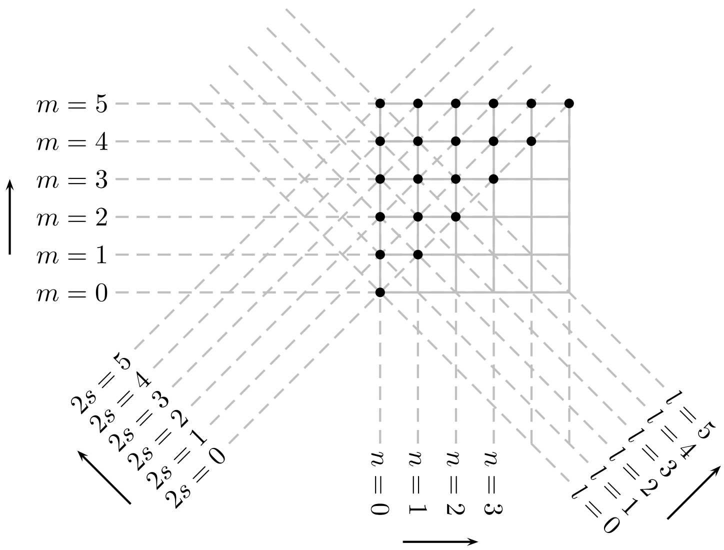

Figure 4.1: Double summation.

Show that in terms of l and s the double summation can be expressed as

$$\frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty}\sum_{s} t^l \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{4.61}$$

where the limits on the sum in s are dictated by Eqs. (4.60) depending on l being even or odd. Thus, read out the polynomial expression for Legendre polynomials of degree l to be

$$P_l(x) = \sum_{s} \frac{(l+2s)!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!(2s)!2^l} x^{2s}(-1)^{\frac{l}{2}-s}, \tag{4.62}$$

where the summation on s depends on whether l is even or odd.

(c) Show that

$$\left(\frac{d}{dx}\right)^l x^{l+2s} = \frac{(l+2s)!}{(2s)!}x^{2s}. \tag{4.63}$$

Thus, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} x^{l+2s}(-1)^{\frac{l}{2}-s}. \tag{4.64}$$

(d) For even l the summation in s runs from $s=0$ to $s=l/2$, Thus, writing $l+2s = 2[l-\left(\frac{l}{2}-s\right)]$, show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l}{2}} \frac{l!}{\left(\frac{l}{2}+s\right)!\left(\frac{l}{2}-s\right)!} (x^2)^{l-\left(\frac{l}{2}-s\right)}(-1)^{\left(\frac{l}{2}-s\right)}. \tag{4.65}$$

Then, substituting

$$\frac{l}{2} - s = n, \tag{4.66}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l}{2}} \frac{l!}{(l-n)!n!} (x^2)^{l-n}(-1)^n. \tag{4.67}$$

Note that the summation on n runs from $n = 0$ to $n = l/2$. If we were to extend this sum to $n = l$ verify that the additional terms will have powers in x less than l. Since the terms in the sum are acted upon by l derivatives with respect to x these additional terms will not contribute. Thus, show that

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}. \tag{4.68}$$

Similarly, for odd l the summation is s runs as

$$2s = 1, 3, 5, \ldots, l, \tag{4.69}$$

or

$$\frac{2s-1}{2} = 0, 1, 2, \ldots, \frac{l-1}{2}. \tag{4.70}$$

Thus, substituting

$$s' = \frac{2s-1}{2} = s - \frac{1}{2}, \tag{4.71}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{s=0}^{\frac{l-1}{2}} \frac{l!}{\left(\frac{l+1}{2}+s\right)!\left(\frac{l-1}{2}-s\right)!} x^{l+1+2s}(-1)^{\left(\frac{l-1}{2}-s\right)}. \tag{4.72}$$

Substituting

$$\frac{l-1}{2} - s = n \tag{4.73}$$

and writing

$$\frac{l+1}{2} + s = l - \left(\frac{l-1}{2} - s\right) \tag{4.74}$$

show that

$$P_l(x) = \frac{1}{l!\,2^l}\left(\frac{d}{dx}\right)^l \sum_{n=0}^{\frac{l-1}{2}} \frac{l!}{(l-n)!n!}(x^2)^{l-n}(-1)^n. \tag{4.75}$$

Again, like in the case of even l we can extend the sum on n beyond $n = (l-1)/2$, because they do not survive under the action of l derivatives with respect to x. Thus, again, we have

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{l!\,2^l}, \tag{4.76}$$

which is exactly the form obtained for even l. The expression in Eq. (4.76) is the Rodrigues formula for generating the Legendre polynomials of degree l.

4. (**20 points.**) (**Orthogonality relations**.)
   Refer 2022Nov28.

4.2.1 Problems

1. (**20 points.**) Using Mathematica (or another graphing tool) plot the Legendre polynomials $P_l(x)$ for $l = 0, 1, 2, 3, 4$ on the same plot. Note that $-1 \leq x \leq 1$. Based on the pattern you see what can you conclude about the number of roots for $P_l(x)$. In Mathematica these plots are generated using the following commands:
   `Plot[{LegendreP[0,x], LegendreP[1,x], LegendreP[2,x], LegendreP[3,x],`
   `LegendreP[4,x] },{x,-1,1}]`
   Compare your plots with those in Wikipedia article on ‘Legendre Polynomials’. While there read the Wikipedia article on Adrien-Marie Legendre and the associated ‘Portrait Debacle’.

2. (**20 points.**) Legendre polynomials are conveniently generated using the relation

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}, \tag{4.77}$$

where $-1 \leq x \leq 1$. Evaluate Legendre polynomials of degree $l = 0, 1, 2, 3, 4$ in this manner.

3. (**20 points.**) Legendre polynomials $P_l(x)$ satisfy the relation

$$\int_{-1}^{1} dx\, P_l(x) = 0 \quad \text{for} \quad l \geq 1. \tag{4.78}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

4. (**20 points.**) Legendre polynomials satisfy the differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + l(l+1)\right] P_l(\cos\theta) = 0. \tag{4.79}$$

Verify this explicitly for $l = 0, 1, 2, 3, 4$.

5. (**20 points.**) Legendre polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} dx\, P_l(x) P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{4.80}$$

Verify this explicitly for $l = 0, 1, 2$ and $l' = 0, 1, 2$. The orthogonality relation is also expressed as

$$\int_0^{\pi} \sin\theta d\theta\, P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{2l+1}\delta_{ll'}. \tag{4.81}$$

6. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(x) P_l(x') = \delta(x - x'). \tag{4.82}$$

This is for your information. No work needed. The completeness relation is also expressed as

$$\sum_{l=0}^{\infty} \frac{2l+1}{2} P_l(\cos\theta) P_l(\cos\theta') = \frac{\delta(\theta-\theta')}{\sin\theta}. \tag{4.83}$$

7. (**Example.**) The Legendre polynomials of order l are

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{4.84}$$

In particular,

$$P_0(x) = 1, \tag{4.85a}$$

$$P_1(x) = x, \tag{4.85b}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{4.85c}$$

The expansion

$$F(x,t) = \frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{l=0}^{\infty} t^l P_l(x), \qquad |t| < 1, \tag{4.86}$$

is usually referred to as the generating function for Legendre's polynomials. From it all the properties of these polynomials may be derived.

8. (**Example.**) The Legendre polynomials of order l satisfy the recurrence relation

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x), \qquad l = 1, 2, 3, \ldots. \tag{4.87}$$

Recall,

$$P_0(x) = 1, \tag{4.88a}$$
$$P_1(x) = x. \tag{4.88b}$$

Derive the explicit expression for $P_4(x)$ using the recurrence relation.

9. (**20 points.**) Express the function

$$\sigma(\theta) = \cos^2\theta \tag{4.89}$$

in terms of Legendre polynomials.
Solution:

$$\sigma(\theta) = \frac{2}{3}P_2(\cos\theta) + \frac{1}{3}P_0(\cos\theta). \tag{4.90}$$

10. (**20 points.**) Express the function

$$\sigma(\theta) = \cos 2\theta \tag{4.91}$$

in terms of Legendre polynomials.
Solution:

$$\sigma(\theta) = \frac{4}{3}P_2(\cos\theta) - \frac{1}{3}P_0(\cos\theta). \tag{4.92}$$

11. (**20 points.**) Legendre polynomials satisfy the completeness relation

$$\sum_{l=0}^{n} P_l(\cos\theta)P_{n-l}(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{4.93}$$

Verify this explicitly for $l = 0, 1, 2$. Prove this for arbitrary n. No work needed. I have still not attempted on it.

12. (**20 points.**) The generating function for the Legendre polynomials $P_l(x)$ of degree l is

$$g(t,x) = \frac{1}{\sqrt{1+t^2-2xt}} = \sum_{l=0}^{\infty} t^l P_l(x). \tag{4.94}$$

Evaluate $P_{11}(0)$ and $P_{12}(0)$.

4.3 Electric potential of $2l$-pole

1. (**10 points.**) The surface charge density on the surface of a charged sphere is given by

$$\sigma(\theta,\phi) = \frac{Q}{4\pi a^2}\cos^2\theta, \tag{4.95}$$

where θ is the polar angle in spherical coordinates. Express this charge distribution in terms of the Legendre polynomials. Recall,

$$P_0(\cos\theta) = 1, \tag{4.96a}$$
$$P_1(\cos\theta) = \cos\theta, \tag{4.96b}$$
$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}. \tag{4.96c}$$

2. (**10 points.**) The induced charge on the surface of a spherical conducting shell of radius a due to a point charge q placed a distance b away from the center is given by

$$\rho(\mathbf{r}) = \sigma(\theta,\phi)\,\delta(r-a), \tag{4.97}$$

where

$$\sigma(\theta,\phi) = -\frac{q}{4\pi a}\frac{(r_>^2 - r_<^2)}{(a^2+b^2-2ab\cos\theta)^{\frac{3}{2}}}, \tag{4.98}$$

where $r_< = \text{Min}(a,b)$ and $r_> = \text{Max}(a,b)$. Calculate the dipole moment of this charge configuration (excluding the original charge q) using

$$\mathbf{d} = \int d^3r\,\mathbf{r}\,\rho(\mathbf{r}), \tag{4.99}$$

for the two cases $a < b$ and $a > b$, representing the charge being inside or outside the sphere. (Hint: First complete the r integral and the ϕ integral. Then, for the θ integral substitute $a^2+b^2-2ab\cos\theta = y$.)

3. (**20 points.**) Consider the electric potential due to a solid sphere with uniform charge density Q. The angular integral in this evaluation involves the integral

$$\frac{1}{2}\int_{-1}^{1} dt \frac{1}{\sqrt{r^2+{r'}^2-2rr't}}. \tag{4.100}$$

Evaluate the integral for $r < r'$ and $r' < r$, where r and r' are distances measured from the center of the sphere. (Hint: Substitute $r^2+{r'}^2-2rr't = y$.)

4. (**20 points.**) Recollect Legendre polynomials of order l

$$P_l(x) = \left(\frac{d}{dx}\right)^l \frac{(x^2-1)^l}{2^l l!}. \tag{4.101}$$

In particular

$$P_0(x) = 1, \tag{4.102a}$$
$$P_1(x) = x, \tag{4.102b}$$
$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}. \tag{4.102c}$$

Consider a charged spherical shell of radius a consisting of a charge distribution in the polar angle alone,

$$\rho(\mathbf{r}') = \sigma(\theta')\,\delta(r'-a). \tag{4.103}$$

The electric potential *on the z-axis*, $\theta = 0$ and $\phi = 0$, is then given by

$$\begin{aligned}\phi(r,0,0) &= \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ &= \frac{2\pi a^2}{4\pi\varepsilon_0}\int_0^\pi \sin\theta' d\theta' \frac{\sigma(\theta')}{\sqrt{r^2+a^2-2ar\cos\theta'}},\end{aligned} \tag{4.104}$$

after evaluating the r' and ϕ' integral.

(a) Consider a uniform charge distribution on the shell,

$$\sigma(\theta) = \frac{Q}{4\pi a^2}P_0(\cos\theta). \tag{4.105}$$

Evaluate the integral in Eq. (4.104) to show that

$$\phi(r,0,0) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r_>}, \tag{4.106}$$

where $r_< = \text{Min}(a,r)$ and $r_> = \text{Max}(a,r)$.
Note: This was done in class. Nevertheless, present the relevant steps.

(b) Next, consider a (pure dipole, 2×1-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_1(\cos\theta). \tag{4.107}$$

Evaluate the integral in Eq. (4.104) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{3} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right). \tag{4.108}$$

Note: This was done in class. Nevertheless, present the relevant steps.

(c) Next, consider a (pure quadrapole, 2×2-pole,) charge distribution of the form,

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_2(\cos\theta). \tag{4.109}$$

Evaluate the integral in Eq. (4.104) to show that

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{5} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^2. \tag{4.110}$$

(d) For a (pure $2l$-pole) charge distribution

$$\sigma(\theta) = \frac{Q}{4\pi a^2} P_l(\cos\theta) \tag{4.111}$$

the integral in Eq. (4.104) leads to

$$\phi(r, 0, 0) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{(2l+1)} \frac{1}{r_>} \left(\frac{r_<}{r_>}\right)^l. \tag{4.112}$$

Note: No work needs to be submitted for this part. We will prove this in class.

5. (**20 points.**) Calculate the dipole moment

$$\mathbf{d} = \int d^3r \, \mathbf{r} \, \rho(\mathbf{r}) \tag{4.113}$$

of a charged spherical shell of radius a with charge density

$$\rho(\mathbf{r}) = \frac{Q}{4\pi a^2} P_1(\cos\theta)\delta(r - a). \tag{4.114}$$

6. (**20 points.**) The surface charge densities on the surface of two separate and independent charged spheres are given by

$$\sigma_1(\theta, \phi) = \frac{Q}{4\pi a^2} \cos\theta, \tag{4.115}$$

$$\sigma_2(\theta, \phi) = \frac{Q}{4\pi a^2} \cos^2\theta, \tag{4.116}$$

where θ is the polar angle in spherical coordinates. Calculate the total charge on each sphere by integrating over the surface of each sphere.

4.4 Multipole expansion

1. (**20 points.**) Consider a configuration of charges $q_1, q_2, q_3, \ldots$, at positions $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots$, and let $\mathbf{r}_0$ be an arbitrary point in space. Define the postion vector of the charges with respect to $\mathbf{r}_0$ to be

$$\mathbf{R}_i = \mathbf{r}_i - \mathbf{r}_0. \tag{4.117}$$

The monopole moment, the dipole moment, and the quadrupole moment of this configuration is given by

$$Q = q_1 + q_2 + q_3 + \ldots, \tag{4.118a}$$
$$\mathbf{d} = q_1\mathbf{R}_1 + q_2\mathbf{R}_2 + q_3\mathbf{R}_3 + \ldots, \tag{4.118b}$$
$$\mathbf{q} = q_1(3\mathbf{R}_1\mathbf{R}_1 - R_1^2\mathbf{1}) + q_2(3\mathbf{R}_2\mathbf{R}_2 - R_2^2\mathbf{1}) + q_3(3\mathbf{R}_3\mathbf{R}_3 - R_3^2\mathbf{1}) + \ldots, \tag{4.118c}$$

respectively. Evaluate the monopole moment, the dipole moment, and the quadrupole moment of three identical charges, each having charge q, positioned on the x axis at a, $2a$, and $3a$, respectively.

2. (**20 points.**) Given the quadrupole tensor

$$\mathbf{q} = q_1(3\mathbf{R}_1\mathbf{R}_1 - R_1^2\mathbf{1}) + q_2(3\mathbf{R}_2\mathbf{R}_2 - R_2^2\mathbf{1}) + q_3(3\mathbf{R}_3\mathbf{R}_3 - R_3^2\mathbf{1}) + \ldots, \tag{4.119}$$

show that

$$\operatorname{tr}\mathbf{q} = 0. \tag{4.120}$$

3. (**20 points.**) The monopole moment, the dipole moment, and the quadrupole moment, of a charge distribution $\rho(\mathbf{r})$ is given by

$$Q = \int d^3r\,\rho(\mathbf{r}), \tag{4.121a}$$
$$\mathbf{d} = \int d^3r\,\rho(\mathbf{r})\,\mathbf{r}, \tag{4.121b}$$
$$\mathbf{q} = \int d^3r\,\rho(\mathbf{r})\left[3\mathbf{r}\mathbf{r} - r^2\mathbf{1}\right], \tag{4.121c}$$

respectively. Consider a charge distribution consisting of a single point charge. If it is placed at the origin calculate the monopole moment, dipole moment, and quadrupole moment, of the charge distribution. Repeat the calculation if the position of the point charge is $(a, 0, 0)$.

4. (**20 points.**) Show that a configuration consisting of three charges with zero electric monopole moment and zero electric dipole moment is collinear.
Hint: Let the three charges be q_1, q_2, and q_3, and their positions be $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. Show that we can express $(\mathbf{r}_1 - \mathbf{r}_3) = a(\mathbf{r}_1 - \mathbf{r}_2)$ and $(\mathbf{r}_2 - \mathbf{r}_3) = b(\mathbf{r}_1 - \mathbf{r}_2)$. Find a and b.

5. (**20 points.**) We have three charges q_1, q_2, and q_3, at positions $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, respectively. If the configuration has zero electric monopole moment and zero electric dipole moment, then show that the three charges are collinear. Further, show that the electric quadrupole moment of the configuration is

$$\mathbf{q} = q_h\Big[3(\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_1 - \mathbf{r}_2) - (\mathbf{r}_1 - \mathbf{r}_2)\cdot(\mathbf{r}_1 - \mathbf{r}_2)\mathbf{1}\Big]. \tag{4.122}$$

where q_h is the harmonic mean of q_1 and q_2 given by

$$\frac{1}{q_h} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{4.123}$$

6. (**20 points.**) Two charges with charge $+q$ and $-q$ are placed at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. Find the monopole moment and the dipole moment of this configuration of two charges. Is the dipole moment independent of the choice of origin? Is the dipole moment independent of the orientation of the coordinate axis?

7. (**20 points.**) Two charges with charge $+q$ each are placed at $(a, 0, 0)$ and $(-a, 0, 0)$. A third charge with charge $-2q$ is placed at the origin. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of the two charges.

8. (**20 points.**) Two electrons and two protons are placed at the corners of a square of length a, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the square in the xy plane?

9. (**20 points.**) Two electrons and two protons are placed at the corners of a rectangle of length a and width b, such that the electrons are at diagonally opposite corners. For simplicity let us choose them to be in the xy plane. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges. Do these moments depend on the orientation of the rectangle in the xy plane?

10. (**20 points.**) A positive charge q is placed at $(a, 0, 0)$. Two negative charges of charge $-q$ each are placed at $(-a/2, a\sqrt{3}/2, 0)$ and $(-a/2, -a\sqrt{3}/2, 0)$. Find the monopole moment, dipole moment, and the quadrupole moment, of this configuration of charges.

11. (**20 points.**) Two charges, each with charge $+q$, are placed at positions $\mathbf{r}_1 = a\,\hat{\mathbf{i}}$ and $\mathbf{r}_2 = a\,\hat{\mathbf{j}}$. A third charge with charge $-2q$ is placed at the origin. Find the monopole moment and the dipole moment of this configuration of three charges.

12. (**20 points.**) Two charges, each with charge $+q$, are placed at positions $\mathbf{r}_1 = a\,\hat{\mathbf{i}}$ and $\mathbf{r}_2 = a\,\hat{\mathbf{j}}$. Another set of two charges, each with charge $-q$, are placed at positions $\mathbf{r}_3 = -a\,\hat{\mathbf{i}}$ and $\mathbf{r}_4 = -a\,\hat{\mathbf{j}}$. Find the monopole moment, the dipole moment, and the quadrupole moment, of this configuration of four charges.

13. (**20 points.**) Evaluate the monopole moment, the dipole moment, and the quadrupole moment of countable infinite identical charges, each having charge q, positioned on the x axis at $a, a/2, a/3, \ldots$, respectively. Hint: Express the moments in terms of the Riemann zeta function $\zeta(s)$, which is well defined and finite for the particular values of s here.

4.5 Electric potential

1. (**40 points.**) Find the electric potential due to a uniformly charged ring of radius a and total charge Q everywhere.

   (a) Let the ring be infinitely thin. Let it be placed on the x-y plane with its center at the origin. Show that the charge density for the ring in spherical coordinates can be expressed in the form

   $$\rho(\mathbf{r}') = \frac{Q}{2\pi a} \frac{\delta\left(\theta' - \frac{\pi}{2}\right)}{r'} \delta(r' - a). \tag{4.124}$$

   Verify that $\int d^3r' \rho(\mathbf{r}') = Q$.

   (b) Using symmetry argue that the electric potential has no dependence in the azimuth angle ϕ. Thus,

   $$\phi(\mathbf{r}) = \phi(r, \theta). \tag{4.125}$$

   We will obtain a solution for the electric potential as an expansion in Legendre polynomials.

   (c) Starting from

   $$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{4.126}$$

   find the solution for the electric potential on the z axis (where $\theta = 0$) to be

   $$\phi(r, 0) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{a^2 + r^2}}. \tag{4.127}$$

Using the binomial expansion

$$\frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} x^{2n} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2} \tag{4.128}$$

express the electric potential on the z axis in the form

$$\phi(r,0) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^{2n} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2}, \tag{4.129}$$

where $r_< = \text{Min}(r,a)$ and $r_> = \text{Max}(r,a)$.

(d) Let the Legendre expansion of the electric potential be

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a} \sum_{l=0}^{\infty} A_l(r) P_l(\cos\theta). \tag{4.130}$$

The electric potential satisfies the Laplacian

$$-\nabla^2\phi = 0 \tag{4.131}$$

for points not on the ring. Using the Laplacian in spherical coordinates and the differential equation satisfied by the Legendre polynomials, deduce the differential equation for the coefficients $A_l(r)$ to be

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] A_l(r) = 0. \tag{4.132}$$

Show that

$$A_l(r) = \alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}. \tag{4.133}$$

Thus, the Legendre expansion for the electric potential is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{a} \sum_{l=0}^{\infty} \left[\alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}\right] P_l(\cos\theta). \tag{4.134}$$

Requiring the boundary condition that the electric potential be zero for $r \to \infty$ and is finite at $r = 0$, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{a} \displaystyle\sum_{l=0}^{\infty} \alpha_l \left(\frac{r}{a}\right)^l P_l(\cos\theta), & r < a, \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r} \displaystyle\sum_{l=0}^{\infty} \beta_l \left(\frac{a}{r}\right)^l P_l(\cos\theta), & a < r. \end{cases} \tag{4.135}$$

(e) Using Eq. (4.135), we have

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{a} \displaystyle\sum_{l=0}^{\infty} \alpha_l \left(\frac{r}{a}\right)^l, & r < a, \\ \dfrac{1}{4\pi\varepsilon_0} \dfrac{Q}{r} \displaystyle\sum_{l=0}^{\infty} \beta_l \left(\frac{a}{r}\right)^l, & a < r. \end{cases} \tag{4.136}$$

where we used $P_l(1) = 1$. Comparing Eqs. (4.129) and (4.136) show that

$$\alpha_l = \beta_l = \begin{cases} 0 & l = 1, 3, 5, \ldots, \\ \dfrac{(-1)^n}{2^{2n}} \dfrac{(2n)!}{(n!)^2}, & l = 2n, \quad n = 0, 1, 2, \ldots. \end{cases} \tag{4.137}$$

Thus, show that

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r_>} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}} \frac{(2n)!}{(n!)^2} \left(\frac{r_<}{r_>}\right)^{2n} P_{2n}(\cos\theta). \tag{4.138}$$

2. (**40 points.**) Let us consider a uniformly charged circular disc of radius a and total charge Q. Let the disc be infinitely thin. Let it be placed on the x-y plane with its center at the origin.

   (a) Show that the charge density for the disc in spherical coordinates can be expressed in the form

$$\rho(\mathbf{r}') = \frac{Q}{\pi a^2} \frac{\delta\left(\theta' - \frac{\pi}{2}\right)}{r'} \theta(a - r'). \tag{4.139}$$

   Verify that $\int d^3r' \rho(\mathbf{r}') = Q$.

   (b) Using symmetry argue that the electric potential has no dependence in the azimuth angle ϕ. Thus,

$$\phi(\mathbf{r}) = \phi(r, \theta). \tag{4.140}$$

   Our goal here will be to obtain a solution for the electric potential as an expansion in Legendre polynomials.

   (c) Starting from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{4.141}$$

   find the solution for the electric potential on the z axis (where $\theta = 0$) to be

$$\phi(r, 0) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a^2} \left[\sqrt{a^2 + r^2} - r\right]. \tag{4.142}$$

   Using the binomial expansion

$$\sqrt{1 + x^2} = 1 + \sum_{n=1}^{\infty} x^{2n} \frac{(-1)^{n-1}}{2n} \frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \tag{4.143}$$

   express the electric potential on the z axis in the form

$$\phi(r, 0) = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a} \left[1 - \frac{r}{a} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n} \frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2} \left(\frac{r}{a}\right)^{2n}\right], & r < a, \\ \frac{1}{4\pi\varepsilon_0} \frac{2Q}{r} \sum_{n=0}^{\infty} \frac{(-1)^n}{2(n+1)} \frac{(2n)!}{2^{2n}(n!)^2} \left(\frac{a}{r}\right)^{2n}, & a < r. \end{cases} \tag{4.144}$$

   (d) Let the Legendre expansion of the electric potential be

$$\phi(r, \theta) = \frac{1}{4\pi\varepsilon_0} \frac{2Q}{a} \sum_{l=0}^{\infty} A_l(r) P_l(\cos\theta). \tag{4.145}$$

   The electric potential satisfies the Laplacian

$$-\nabla^2 \phi = 0 \tag{4.146}$$

   outside the disc. Using the Laplacian in spherical coordinates and the differential equation satisfied by the Legendre polynomials, deduce the differential equation for the coefficients $A_l(r)$ to be

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] A_l(r) = 0. \tag{4.147}$$

   Show that

$$A_l(r) = \alpha_l \left(\frac{r}{a}\right)^l + \beta_l \left(\frac{a}{r}\right)^{l+1}. \tag{4.148}$$

Thus, the Legendre expansion for the electric potential is

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\sum_{l=0}^{\infty}\left[\alpha_l\left(\frac{r}{a}\right)^l + \beta_l\left(\frac{a}{r}\right)^{l+1}\right]P_l(\cos\theta). \tag{4.149}$$

Requiring the boundary condition that the electric potential be zero for $r \to \infty$ and is finite at $r = 0$, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\frac{r}{a}\right)^l P_l(\cos\theta), & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\frac{a}{r}\right)^l P_l(\cos\theta), & a < r. \end{cases} \tag{4.150}$$

(e) Using Eq. (4.150), we have

$$\phi(r,0) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\displaystyle\sum_{l=0}^{\infty}\alpha_l\left(\frac{r}{a}\right)^l, & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{l=0}^{\infty}\beta_l\left(\frac{a}{r}\right)^l, & a < r. \end{cases} \tag{4.151}$$

where we used $P_l(1) = 1$. Comparing Eqs. (4.144) and (4.151) show that

$$\alpha_l = \begin{cases} 1 & l = 0, \\ -1 & l = 1, \\ 0 & l = 3, 5, 7, \ldots, \\ \dfrac{(-1)^{n-1}}{2n}\dfrac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}, & l = 2n, \quad n = 1, 2, 3, \ldots, \end{cases} \tag{4.152}$$

and

$$\beta_l = \begin{cases} 0 & l = 1, 3, 5, \ldots, \\ \dfrac{(-1)^n}{2(n+1)}\dfrac{(2n)!}{2^{2n}(n!)^2} & l = 2n, \quad n = 0, 1, 2, 3, \ldots. \end{cases} \tag{4.153}$$

Thus, show that

$$\phi(r,\theta) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{a}\left[1 - \dfrac{r}{a}P_1(\cos\theta) + \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\frac{[2(n-1)]!}{2^{2(n-1)}[(n-1)!]^2}\left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)\right], & r < a, \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{2Q}{r}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{2(n+1)}\frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta), & a < r. \end{cases} \tag{4.154}$$

(f) For $r \ll a$ the disc should simulate a plate of infinite extent. Show that

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{a}\left[1 - \frac{z}{a}\right] + \mathcal{O}\left(\frac{z}{a}\right)^2, \tag{4.155}$$

using $rP_1(\cos\theta) = z$. This leads to the electric field for a plate of infinite extent,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi = \hat{\mathbf{z}}\frac{\sigma}{2\varepsilon_0}, \tag{4.156}$$

where $\sigma = Q/(\pi a^2)$.

4.6 Inversion

1. (**20 points.**) Inversion is a transformation that maps a point $\mathbf{r}$ inside (outside) a sphere of radius a to a point

$$\mathbf{r}_a = \frac{a^2}{r^2}\mathbf{r} \tag{4.157}$$

outside (inside) the sphere. Given that the function $\phi(\mathbf{r})$ satisfies the Laplacian,

$$\nabla^2\phi(\mathbf{r}) = 0, \tag{4.158}$$

show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) \tag{4.159}$$

also satisfies the Laplacian for $r \neq 0$. That is,

$$\nabla^2\left[\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right)\right] = 0. \tag{4.160}$$

To this end, using Eq. (4.157) evaluate $\mathbf{r}_a\cdot\mathbf{r}_a$ and thus derive

$$r_a r = a^2. \tag{4.161}$$

Then, show that

$$\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right) = \frac{r_a}{a}\,\phi(\mathbf{r}_a). \tag{4.162}$$

To express the gradient in terms of the inverted variable $\mathbf{r}_a$ write

$$\boldsymbol{\nabla} = \frac{\partial}{\partial\mathbf{r}} = \frac{\partial\mathbf{r}_a}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}_a} = (\boldsymbol{\nabla}\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a. \tag{4.163}$$

Show that

$$(\boldsymbol{\nabla}\,\mathbf{r}_a) = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a). \tag{4.164}$$

Thus, show that

$$\boldsymbol{\nabla} = \frac{1}{a^2}(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a \tag{4.165}$$

and

$$\nabla^2 = \frac{1}{a^4}\left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a\right]\cdot\left[(\mathbf{1}\,r_a^2 - 2\,\mathbf{r}_a\,\mathbf{r}_a)\cdot\boldsymbol{\nabla}_a\right]. \tag{4.166}$$

Expand the operations and simplify to derive

$$a^4\nabla^2 = r_a^4\nabla_a^2 - 2r_a^2\mathbf{r}_a\cdot\boldsymbol{\nabla}_a. \tag{4.167}$$

To prove the statement in Eq. (4.160) show that

$$\nabla^2\left[\frac{a}{r}\,\phi\left(\frac{a^2}{r^2}\mathbf{r}\right)\right] = \frac{r_a^5}{a^5}\nabla_a^2\phi(\mathbf{r}_a) = 0. \tag{4.168}$$

4.7 Solid harmonics versus surface harmonics

1. (**20 points.**) The fundamental solution to Laplace's equation is the electric potential due to a point charge,

$$\frac{q}{4\pi\varepsilon_0}\frac{1}{r}. \tag{4.169}$$

Dropping $q/(4\pi\varepsilon_0)$ we have

$$\nabla^2\frac{1}{r} = 0, \quad r \neq 0. \tag{4.170}$$

In terms of this solution, we can generate a large number of others. For example, for constant vectors $\mathbf{s}_1$,

$$\nabla^2\left[(\mathbf{s}_1\cdot\boldsymbol{\nabla})\frac{1}{r}\right] = 0, \tag{4.171}$$

because the gradient operators commute with itself and $\mathbf{s}_1$ is a constant. Solid harmonics of degree $-(l+1)$ are defined as

$$V_l(\mathbf{r}) = \frac{1}{l!}(-\mathbf{s}_1\cdot\boldsymbol{\nabla})(-\mathbf{s}_2\cdot\boldsymbol{\nabla})\ldots(-\mathbf{s}_l\cdot\boldsymbol{\nabla})\frac{1}{r} \tag{4.172}$$

for $l = 1, 2, \ldots,$ with

$$V_0(\mathbf{r}) = \frac{1}{r} \tag{4.173}$$

for $l = 0$. Verify that the solid harmonics satisfy the Laplace equation, that is,

$$\nabla^2 V_l(\mathbf{r}) = 0, \qquad l = 0, 1, 2, \ldots. \tag{4.174}$$

It is insightful to see the explicit form of the solid harmonics after the gradient operations have been evaluated.

(a) Define

$$\mu_i = (\mathbf{s}_i\cdot\hat{\mathbf{r}}), \qquad \tilde{\mu}_i = (\mathbf{s}_i\cdot\mathbf{r}), \tag{4.175a}$$

$$\lambda_{ij} = (\mathbf{s}_i\cdot\mathbf{s}_j), \qquad \tilde{\lambda}_{ij} = (\mathbf{s}_i\cdot\mathbf{s}_j)r^2. \tag{4.175b}$$

Show that

$$(-\mathbf{s}_i\cdot\boldsymbol{\nabla})\tilde{\mu}_j = -\frac{\tilde{\lambda}_{ij}}{r^2}, \tag{4.176a}$$

$$(-\mathbf{s}_i\cdot\boldsymbol{\nabla})\frac{1}{r^m} = \frac{m}{r^{m+2}}\,\tilde{\mu}_i, \tag{4.176b}$$

$$(-\mathbf{s}_k\cdot\boldsymbol{\nabla})\tilde{\lambda}_{ij} = -\frac{2\tilde{\mu}_k\tilde{\lambda}_{ij}}{r^2}. \tag{4.176c}$$

(b) Show that

$$V_1 = \frac{1}{1!}\frac{1}{r^3}\Big[\tilde{\mu}_1\Big], \tag{4.177a}$$

$$V_2 = \frac{1}{2!}\frac{1}{r^5}\Big[3\tilde{\mu}_1\tilde{\mu}_2 - \tilde{\lambda}_{12}\Big], \tag{4.177b}$$

$$V_3 = \frac{1}{3!}\frac{1}{r^7}\Big[15\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3 - 3\tilde{\mu}_1\tilde{\lambda}_{23} - 3\tilde{\mu}_2\tilde{\lambda}_{31} - 3\tilde{\mu}_3\tilde{\lambda}_{12}\Big], \tag{4.177c}$$

$$V_4 = \frac{1}{4!}\frac{1}{r^9}\Big[105\tilde{\mu}_1\tilde{\mu}_2\tilde{\mu}_3\tilde{\mu}_4 - 15\tilde{\mu}_1\tilde{\mu}_2\tilde{\lambda}_{34} - 15\tilde{\mu}_1\tilde{\mu}_3\tilde{\lambda}_{24} - 15\tilde{\mu}_1\tilde{\mu}_4\tilde{\lambda}_{23} - 15\tilde{\mu}_2\tilde{\mu}_3\tilde{\lambda}_{14} - 15\tilde{\mu}_2\tilde{\mu}_4\tilde{\lambda}_{13} - 15\tilde{\mu}_3\tilde{\mu}_4\tilde{\lambda}_{12} + 3\tilde{\lambda}_{12}\tilde{\lambda}_{34} + 3\tilde{\lambda}_{13}\tilde{\lambda}_{24} + 3\tilde{\lambda}_{34}\tilde{\lambda}_{12}\Big]. \tag{4.177d}$$

For bringing compactness we introduce the notation

$$\mu^{l-2m}\lambda^m = \mu_1\mu_2\dots\mu_{l-2m}\lambda_{..}\lambda_{..}\dots + \text{combinations} \tag{4.178}$$

in terms of which we find

$$V_l(\mathbf{r}) = \frac{1}{r^{l+1}}\sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{4.179}$$

(c) Surface (or spherical) harmonics $Y_l(\hat{\mathbf{r}})$ of degree l are defined using the relation

$$V_l(\mathbf{r}) = \frac{Y_l(\hat{\mathbf{r}})}{r^{l+1}}. \tag{4.180}$$

Show that

$$Y_l(\hat{\mathbf{r}}) = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\mu^{l-2m}\lambda^m. \tag{4.181}$$

(d) Inversion between points $\mathbf{r}$ and $\mathbf{r}_a$ about a sphere of radius a is described by the relations

$$\frac{r_a}{a} = \frac{a}{r} \tag{4.182}$$

and

$$\mathbf{r}_a = \frac{a^2}{r^2}\mathbf{r} \tag{4.183}$$

and

$$\mathbf{r} = \frac{a^2}{r_a^2}\mathbf{r}_a. \tag{4.184}$$

Using inversion we conclude that for every solid harmonic $V_l(\mathbf{r})$ that satisfies the Laplacian there exists another solid harmonic

$$U_l(\mathbf{r}) = \frac{a}{r}V_l\left(\frac{a^2}{r^2}\mathbf{r}\right) \tag{4.185}$$

that also satisfies the Laplacian. Show that

$$U_1(\mathbf{r}) = \frac{(\mathbf{s}_1\cdot\mathbf{r})}{a^3}. \tag{4.186}$$

In general show that

$$U_l(\mathbf{r}) = \frac{r^l}{a^{2l+1}}Y_l(\hat{\mathbf{r}}). \tag{4.187}$$

Solid harmonics $H_l(\mathbf{r})$ of degree l are defined using the relation

$$H_l(\mathbf{r}) = a^{2l+1}U_l(\mathbf{r}) = r^lY_l(\hat{\mathbf{r}}). \tag{4.188}$$

Show that

$$H_l = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^{l-m}l!(l-m)!}(-1)^m\tilde{\mu}^{l-2m}\tilde{\lambda}^m. \tag{4.189}$$

(e) Zonal harmonics $P_l(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})$ of order l are defined to be surface harmonics of degree l with the special choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \hat{\mathbf{z}}. \tag{4.190}$$

Then, $\lambda_{ij} = 1$ and all μ_i's are identical, say $\mu = (\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}) = \cos\theta$. Show that

$$P_l(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}) = \sum_{m=0}^{l-1}\frac{(2[l-m])!}{2^l m!(l-m)!(l-2m)!}(-1)^m(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})^{l-2m}. \tag{4.191}$$

Recognize that the zonal harmonics $P_l(\cos\theta)$ are the Legendre polynomials $P_l(\cos\theta)$.

(f) Spherical harmonics $Y_l(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})$ of order l are defined to be surface harmonics of degree l with the special choice

$$\mathbf{s}_1 = \mathbf{s}_2 = \cdots = \mathbf{s}_l = \hat{\mathbf{z}}. \tag{4.192}$$

$$i \tag{4.193}$$

4.8 Spherical harmonics

1. (**40 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta,\phi)$ as a function of θ and ϕ. In particular,

   (a) Plot $\text{Re}\big[Y_{73}(\theta,\phi)\big]$.

   (b) Plot $\text{Im}\big[Y_{73}(\theta,\phi)\big]$.

   (c) Plot $\text{Abs}\big[Y_{73}(\theta,\phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a l and m, and Re or Im or Abs.

   Hint: In Mathematica these plots are generated using the following commands:
   ```
   SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ ϕ,0,2 Pi}]
   ```
   Refer to diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.

2. (**20 points.**) Using the definition of spherical harmonics

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}, \tag{4.194}$$

   evaluate the explicit expressions for $Y_{21}(\theta,\phi)$ and $Y_{2,-2}(\theta,\phi)$.

3. (**30 points.**) Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta,\phi)$ for $l = 0,1,2$, by completing the $l-m$ differentiations in Eq. (4.201). Use the result in Eq. (4.203) to reduce the work by about half.

4. (**20 points.**) The spherical harmonics are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\left(\frac{e^{i\phi}}{\sin\theta}\right)^m\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{4.195}$$

   Express $Y_{l,-l}(\theta,\phi)$ in simplified form.

5. (**20 points.**) The spherical harmonics are given by

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\left(\frac{e^{i\phi}}{\sin\theta}\right)^m\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{4.196}$$

   Express $Y_{ll}(\theta,\phi)$ is terms of l, ϕ and $\sin\theta$.

4.8.1 Generating function: Null vectors

1. (**20 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{4.197}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta), \tag{4.198}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{4.199}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{4.200}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{4.201}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{4.202}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta\to-\theta$, $\phi\to-\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{4.203}$$

2. (**20 points.**) An example of a null-vector is

$$\mathbf{a} = (1, i, 0). \tag{4.204}$$

Construct Y_{ll}. **Incomplete.**

3. (**20 points.**) An example of a null-vector is

$$\mathbf{a} = (-i\cos\alpha, -i\sin\alpha, 1). \tag{4.205}$$

(a) Identify the corresponding $y_\pm$ in Eq. (4.199) to show that, now, ψ_{lm} in Eq. (4.197) is

$$\psi_{lm} = \frac{e^{-im\left(\alpha-\frac{\pi}{2}\right)}}{\sqrt{(l+m)!(l-m)!}}. \tag{4.206}$$

(b) Then, integrate Eq. (4.197) to derive an integral representation for spherical harmonics,

$$\frac{1}{l!}\int_0^{2\pi}\frac{d\alpha}{2\pi}\,e^{im\alpha}\big[\cos\theta - i\sin\theta\cos(\phi-\alpha)\big]^l = \sqrt{\frac{4\pi}{2l+1}}\frac{i^m\,Y_{lm}(\theta,\phi)}{\sqrt{(l+m)!(l-m)!}}. \tag{4.207}$$

(c) By setting $m=0$ derive the corresponding integral representation for Legendre polynomial $P_l(\cos\theta)$:

$$\int_0^{\pi}\frac{d\alpha}{\pi}\big[\cos\theta - i\sin\theta\cos\alpha\big]^l = P_l(\cos\theta). \tag{4.208}$$

(d) Use the integral representation for $J_0(t)$,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}, \tag{4.209}$$

to show that

$$P_l(\cos\theta) = \left(\cos\theta - \sin\theta \frac{d}{dt}\right)^l J_0(t)\bigg|_{t=0}. \tag{4.210}$$

Verify this for $l = 0, 1, 2$.

(e) Now let $\theta = x/l$ and, for fixed x, consider the limit $l \to \infty$, to obtain

$$\lim_{l\to\infty} P_l\left(\cos\frac{x}{l}\right) = J_0(x), \tag{4.211}$$

which is often used in the approximate form

$$\theta \ll 1, l \gg 1: \qquad P_l(\cos\theta) \sim J_0(l\theta). \tag{4.212}$$

(f) For what geometrical reason does one expect an asymptotic connection between spherical and cylindrical coordinate functions?

4. (**20 points.**) An integral representation for Legendre polynomial $P_l(\cos\theta)$ is

$$P_l(\cos\theta) = \int_0^{\pi} \frac{d\alpha}{\pi} \big[\cos\theta - i\sin\theta\cos\alpha\big]^l. \tag{4.213}$$

(a) Use the integral representation for $J_0(t)$,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}, \tag{4.214}$$

to show that

$$P_l(\cos\theta) = \left(\cos\theta - \sin\theta \frac{d}{dt}\right)^l J_0(t)\bigg|_{t=0}. \tag{4.215}$$

Verify this for $l = 0, 1, 2$.

(b) Now let $\theta = x/l$ and, for fixed x, consider the limit $l \to \infty$, to obtain

$$\lim_{l\to\infty} P_l\left(\cos\frac{x}{l}\right) = J_0(x), \tag{4.216}$$

which is often used in the approximate form

$$\theta \ll 1, l \gg 1: \qquad P_l(\cos\theta) \sim J_0(l\theta). \tag{4.217}$$

(c) For what geometrical reason does one expect an asymptotic connection between spherical and cylindrical coordinate functions? (Hint: Green's function for planar geometry can be written in terms of J_m.)

4.8.2 Differential equation

1. (**20 points.**) Given

$$\left(\frac{a}{r} + \frac{\partial}{\partial r}\right)\left(\frac{b}{r} + \frac{\partial}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{4.218}$$

Find the numbers a and b.

2. (**20 points.**) Polynomials $(\mathbf{a}\cdot\mathbf{r})^l$ of degree l satisfy the Laplacian when $\mathbf{a}$ is a null-vector, that is,

$$(\mathbf{a}\cdot\mathbf{a}) = 0. \tag{4.219}$$

   (a) Show that

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = l(l-1)(\mathbf{a}\cdot\mathbf{r})^{(l-2)}(\mathbf{a}\cdot\mathbf{a}), \tag{4.220}$$

   and conclude

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = 0. \tag{4.221}$$

   (b) Write the polynomial construction in the form

$$(\mathbf{a}\cdot\mathbf{r})^l = r^l(\mathbf{a}\cdot\hat{\mathbf{r}})^l. \tag{4.222}$$

   Observe that $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ has no radial dependence. Thus, in this form, the radial and angular dependence is separated. Starting from the Laplacian in spherical polar coordinates,

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\mathbf{r})^l = 0, \tag{4.223}$$

   deduce

$$\frac{r^l}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l + (\mathbf{a}\cdot\hat{\mathbf{r}})^l\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = 0. \tag{4.224}$$

   (c) Show that

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}r^l = l(l+1)\frac{r^l}{r^2}. \tag{4.225}$$

   Thus, derive the differential equation for the generating function

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right](\mathbf{a}\cdot\hat{\mathbf{r}})^l = 0. \tag{4.226}$$

   (d) Use the generating function

$$\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l}\psi_{lm}\sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi) \tag{4.227}$$

   written in terms of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{4.228}$$

   to derive the differential equation for spherical harmonics

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right]Y_{lm}(\theta,\phi) = 0. \tag{4.229}$$

3. (**20 points.**) For a constant vector $\mathbf{p}$, (without invoking the Maxwell equations,) evaluate

$$\nabla^2\left(\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) \tag{4.230}$$

for $r \neq 0$.
Hints: For insight, recall that the electric potential of a point dipole $\mathbf{p}$ placed at the origin is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}. \tag{4.231}$$

4. (**20 points.**) Show that

$$(\mathbf{a}\cdot\mathbf{r})^l \tag{4.232}$$

satisfies the Laplacian if $\mathbf{a}$ is a null-vector,

$$(\mathbf{a}\cdot\mathbf{a}) = 0. \tag{4.233}$$

(a) That is,

$$\nabla^2(\mathbf{a}\cdot\mathbf{r})^l = 0. \tag{4.234}$$

(b) Show that

$$\nabla^2\left[\frac{1}{r}\left(\mathbf{a}'\cdot\frac{\mathbf{r}}{r^2}\right)^l\right] = 0. \tag{4.235}$$

Note that this does not require $\mathbf{a}'$ to be a null-vector.
Hints: Show that

$$\boldsymbol{\nabla}\left[\frac{(\mathbf{a}\cdot\mathbf{r})^l}{r^{2l+1}}\right] = \frac{(\mathbf{a}\cdot\mathbf{r})^{l-1}}{r^{2l+1}}\left[l\,\mathbf{a} - (2l+1)\frac{\mathbf{a}\cdot\mathbf{r}\mathbf{r}}{r^2}\right]. \tag{4.236}$$

Then, show that

$$\nabla^2\left[\frac{1}{r}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r^2}\right)^l\right] = l(\mathbf{a}\cdot\mathbf{r})^{l-1}\mathbf{a}\cdot\boldsymbol{\nabla}\frac{1}{r^{2l+1}} - \frac{(2l+1)}{r^{2l+3}}\mathbf{r}\cdot\boldsymbol{\nabla}(\mathbf{a}\cdot\mathbf{r})^l - (2l+1)(\mathbf{a}\cdot\mathbf{r})^l\boldsymbol{\nabla}\cdot\frac{\mathbf{r}}{r^{2l+3}}. \tag{4.237}$$

4.8.3 Multipole expansion using spherical harmonics

1. (**20 points.**) Verify that the right hand side of

$$(-\mathbf{r}'\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{\mathbf{r}'\cdot\mathbf{r}}{r^3} \tag{4.238}$$

is a solution to Laplace's equation for $\mathbf{r}\neq 0$. Further, verify the relations

$$\begin{aligned}
(-\mathbf{r}_1'\cdot\boldsymbol{\nabla})(-\mathbf{r}_2'\cdot\boldsymbol{\nabla})\frac{1}{r} &= \frac{\left[3(\mathbf{r}\cdot\mathbf{r}_1')(\mathbf{r}_2'\cdot\mathbf{r}) - (\mathbf{r}_1'\cdot\mathbf{r}_2')(\mathbf{r}\cdot\mathbf{r})\right]}{r^5}, && (4.239\text{a})\\
&= \frac{\mathbf{r}\cdot\left[3\,\mathbf{r}_1'\mathbf{r}_2' - (\mathbf{r}_1'\cdot\mathbf{r}_2')\,\mathbf{1}\right]\cdot\mathbf{r}}{r^5}, && (4.239\text{b})\\
&= \frac{\mathbf{r}_1'\cdot\left[3\,\mathbf{r}\,\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\,\mathbf{1}\right]\cdot\mathbf{r}_2'}{r^5}, && (4.239\text{c})
\end{aligned}$$

which also satisfy Laplace's equation for $\mathbf{r}\neq 0$, but need not be verified here.

2. (**20 points.**) Consider charge Q uniformly distributed on a spherical shell of radius R.

(a) Calculate the dipole moment of this charge distribution about the center of the shell.

(b) Calculate the dipole moment of this charge distribution about the point $\mathbf{r}_0 = \frac{R}{2}\,\hat{\mathbf{k}}$.

(c) What is the analog of the dipole moment in gravity?

(d) Rate of change in quadruple moment of the source of gravity emits gravitational waves. Envisage (simplest possible) mass distributions that could emit gravitational waves.

4.8.4 Orthonormality conditions for spherical harmonics

1. (**20 points.**) For a null-vector $\mathbf{a}$, that satisfies

$$\mathbf{a}\cdot\mathbf{a} = 0, \tag{4.240}$$

the polynomial $(\mathbf{a}\cdot\hat{\mathbf{r}})^l$ of degree l is the generating function of spherical harmonics $Y_{lm}(\theta,\phi)$. To derive the orthonormality properties of spherical harmonics let us consider the product of two generating functions, with null-vectors $\mathbf{a}$ and $\mathbf{a}^*$, integrated over all the angles,

$$\int d\Omega\,(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}, \tag{4.241}$$

where

$$d\Omega = \sin\theta d\theta d\phi. \tag{4.242}$$

(a) After integration over the angles the product of the two generating functions is a scalar. Thus, it has to be constructed out of $(\mathbf{a}\cdot\mathbf{a})$, $(\mathbf{a}^*\cdot\mathbf{a}^*)$, and $(\mathbf{a}^*\cdot\mathbf{a})$. Since $(\mathbf{a}\cdot\mathbf{a})=0$ and $(\mathbf{a}^*\cdot\mathbf{a}^*)=0$, the integral has to be constructed out of $(\mathbf{a}^*\cdot\mathbf{a})$. This is possible only if $l=l'$. Together, we conclude

$$\int d\Omega\,(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'} = \delta_{ll'}(\mathbf{a}^*\cdot\mathbf{a})^l C_l, \tag{4.243}$$

in terms of arbitrary constant C_l.

(b) To determine C_l choose

$$\mathbf{a} = (1,i,0). \tag{4.244}$$

For this choice of null-vector, evaluate $\mathbf{a}^* = (1,-i,0)$, $(\mathbf{a}\cdot\hat{\mathbf{r}}) = \sin\theta e^{i\phi}$, $(\mathbf{a}^*\cdot\hat{\mathbf{r}}) = \sin\theta e^{-i\phi}$, and $(\mathbf{a}^*\cdot\hat{\mathbf{a}}) = 2$. Thus, find

$$C_l = \frac{4\pi}{2^l}\int_0^1 dt(1-t^2)^l, \tag{4.245}$$

after substituting $\cos\theta = t$. Evaluate

$$C_0 = 4\pi. \tag{4.246}$$

Integrate by parts in the integral for C_l to derive the recurrence relation

$$C_l = \frac{l}{2l+1}C_{l-1}. \tag{4.247}$$

Evaluate

$$C_l = \frac{4\pi 2^l l! l!}{(2l+1)!}. \tag{4.248}$$

Thus, conclude

$$\int d\Omega\,\frac{(\mathbf{a}^*\cdot\hat{\mathbf{r}})^l}{l!}\frac{(\mathbf{a}\cdot\hat{\mathbf{r}})^{l'}}{l!} = \delta_{ll'}4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!}. \tag{4.249}$$

(c) For null-vectors constructed out of $y_\pm$ in the form

$$\mathbf{a} = \left(\frac{y_-^2-y_+^2}{2}, \frac{y_-^2+y_+^2}{2i}, y_+y_-\right) \tag{4.250}$$

show that

$$4\pi\frac{(\mathbf{a}^*\cdot\mathbf{a})^l 2^l}{(2l+1)!} = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\psi_{lm}^*\psi_{l'm'}\delta_{mm'}, \tag{4.251}$$

where

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}}. \tag{4.252}$$

Using the generating function

$$\frac{(\mathbf{a}^* \cdot \hat{\mathbf{r}})^l}{l!} = \sum_{m=-l}^{l} \psi_{lm} \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi) \tag{4.253}$$

show that

$$\sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi)$$
$$= \delta_{ll'} \sqrt{\frac{4\pi}{2l+1}} \sqrt{\frac{4\pi}{2l'+1}} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} \psi_{lm}^* \psi_{l'm'} \delta_{mm'}. \tag{4.254}$$

Thus, comparing the two sides of the equality, read out the orthonormality condition for the spherical harmonics,

$$\int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}. \tag{4.255}$$

2. (**20 points.**) Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2-1)^l}{2^l l!}. \tag{4.256}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (4.256).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{4.257}$$

   (c) Using the orthonormality condition for the spherical harmonics

$$\int d\Omega\, Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'} \tag{4.258}$$

   recognize the orthogonality statement for Legendre polynomials,

$$\frac{1}{2} \int_{-1}^{1} dt\, P_l(t) P_{l'}(t) = \frac{\delta_{ll'}}{2l+1}. \tag{4.259}$$

   Use

$$P_0(t) = 1, \quad P_1(t) = t, \quad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \tag{4.260}$$

   to check this explicitly for $l, l' = 0, 1, 2$.

3. (**20 points.**) The spherical harmonics

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \left(\frac{e^{i\phi}}{\sin\theta}\right)^m \left(\frac{d}{d\cos\theta}\right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!} \tag{4.261}$$

satisfy the orthonormality condition

$$\int d\Omega\, Y_{lm}^*(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'}. \tag{4.262}$$

Using the relation between the Legendre's polynomial $P_l(x)$ and the spherical harmonics,

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\theta,\phi), \tag{4.263}$$

derive the orthonormality condition satisfied by Legendre's polynomials.

Chapter 5

Conservation laws

5.1 Flux

1. (**Comment.**) In fluid dynamics a conservation equation (including the dissipative term) has the form

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} + s = 0, \tag{5.1}$$

where ρ, $\mathbf{j}$, and s, are functions of position and time. In this equation the quantity $\mathbf{j}$ is defined as the flux of the quantity ρ. That is, $\mathbf{j}$ represents the flow rate of ρ per unit area. Inadvertently, in vector calculus, the surface integral of a vector field $\mathbf{E}$ over a surface S is also defined as the flux Φ_E of the vector field,

$$\Phi_E = \int_S d\mathbf{a} \cdot \mathbf{E}. \tag{5.2}$$

This is confusing because the conservation equation and surface integral appear in tandem in electrodynamics. For example, Eq. (5.1) with $s = 0$,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{5.3}$$

is the conservation equation for electric charge density ρ, with the current density $\mathbf{j}$ here interpreted as the flux of charge density. However, Eq. (5.2) for the case of current density $\mathbf{j}$ can be written as

$$I = \int_S d\mathbf{a} \cdot \mathbf{j}, \tag{5.4}$$

where I is the current. Thus, current I is the flux of current density $\mathbf{j}$ (in vector calculus), while the current density $\mathbf{j}$ is the flux of charge density ρ (in fluid dynamics context).

5.2 Conservation laws

1. (**60 points.**) When magnetic charges ρ_m and magnetic currents $\mathbf{j}_m$ are permitted, in addition to electric charges ρ_e and electric currents $\mathbf{j}_e$, the Maxwell equations are

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_e, \tag{5.5a}$$
$$\boldsymbol{\nabla} \cdot \mathbf{B} = \rho_m, \tag{5.5b}$$
$$-\boldsymbol{\nabla} \times \mathbf{E} - \frac{\partial}{\partial t}\mathbf{B} = \mathbf{j}_m, \tag{5.5c}$$
$$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial}{\partial t}\mathbf{D} = \mathbf{j}_e, \tag{5.5d}$$

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{5.6a}$$
$$\mathbf{B} = \mu_0 \mathbf{H}. \tag{5.6b}$$

The Lorentz force, in SI units, on an object with electric charge q_e and magnetic charge q_m is

$$\mathbf{F} = q_e \mathbf{E} + q_e \mathbf{v} \times \mathbf{B} + q_m \mathbf{H} - q_m \mathbf{v} \times \mathbf{D}. \tag{5.7}$$

Note the negative sign in the fourth term of the Lorentz force. This is postulated based on the observation that Maxwell equations are symmetric under the replacement

$$\rho_e \to \rho_m, \qquad \rho_m \to -\rho_e, \tag{5.8a}$$
$$\mathbf{j}_e \to \mathbf{j}_m, \qquad \mathbf{j}_m \to -\mathbf{j}_e, \tag{5.8b}$$
$$\mathbf{E} \to \mathbf{H}, \qquad \mathbf{H} \to -\mathbf{E}, \tag{5.8c}$$

which is a special case of duality transformation. The corresponding force density (force per unit volume) $\mathbf{f}$ is

$$\mathbf{f} = \rho_e \mathbf{E} + \mathbf{j}_e \times \mathbf{B} + \rho_m \mathbf{H} - \mathbf{j}_m \times \mathbf{D}. \tag{5.9}$$

The speed of light in vacuum c satisfies the relation

$$c^2 \varepsilon_0 \mu_0 = 1. \tag{5.10}$$

The duality transformation did entice us to consider magnetic monopoles. However, the purpose for introducing magnetic monopoles here is also to gain insight for the sources for the electromagnetic energy density and electromagnetic momentum density as suggested by the associated conservation laws deduced from the Maxwell equations. At any stage of our calculation we can get the standard electrodynamics by switching off the contributions from magnetic charges and currents, by setting $\rho_m = 0$ and $\mathbf{j}_m = 0$.

(a) Conservation of charge: Take the divergence of Ampère's law in Eq. (5.5d), and then use Gauss law for electric field in Eq. (5.5a) to deduce

$$\frac{\partial}{\partial t}\rho_e + \boldsymbol{\nabla} \cdot \mathbf{j}_e = 0. \tag{5.11}$$

This is the statement of conservation of electric charge. Similarly, take the divergence of Faraday's law in Eq. (5.5c), and then use Gauss law for magnetic field in Eq. (5.5b) to deduce

$$\frac{\partial}{\partial t}\rho_m + \boldsymbol{\nabla} \cdot \mathbf{j}_m = 0. \tag{5.12}$$

This is the statement of conservation of magnetic charge.

(b) Conservation of energy: The rate of energy transfer from the electromagnetic field to the charge, the power, is given by

$$\mathbf{F} \cdot \mathbf{v} = q_e \mathbf{v} \cdot \mathbf{E} + q_m \mathbf{v} \cdot \mathbf{H}. \tag{5.13}$$

For a continuous charge distribution, then, the rate of energy transfer from the electromagnetic field to charge distributions is given by

$$\mathbf{j}_e \cdot \mathbf{E} + \mathbf{j}_m \cdot \mathbf{H}. \tag{5.14}$$

Use Ampère's law in Eq. (5.5d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (5.5c) to replace $\mathbf{j}_m$, in Eq. (5.14), to obtain the statement of conservation of energy

$$\frac{\partial}{\partial t}U + \boldsymbol{\nabla} \cdot \mathbf{S} + \mathbf{j}_e \cdot \mathbf{E} + \mathbf{j}_m \cdot \mathbf{H} = 0, \tag{5.15}$$

where

$$U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B} \tag{5.16}$$

is the electromagnetic field energy density and

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} \tag{5.17}$$

is the flux of electromagnetic field energy density (the Poynting vector).
Hints: Use the identities

$$\mathbf{C}\cdot\frac{\partial}{\partial t}\mathbf{C} = \frac{\partial}{\partial t}\left(\frac{C^2}{2}\right) \tag{5.18}$$

for any vector $\mathbf{C}$, and

$$\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) = (\boldsymbol{\nabla}\times\mathbf{E})\cdot\mathbf{H} - (\boldsymbol{\nabla}\times\mathbf{H})\cdot\mathbf{E}. \tag{5.19}$$

(c) Conservation of momentum: We start from the expression for the force density in Eq. (5.9). Use Gauss law for electric and magnetic field in Eqs. (5.5a) and (5.5b) to replace ρ_e and ρ_m, Ampère's law in Eq. (5.5d) to replace $\mathbf{j}_e$, and Faraday's law in Eq. (5.5c) to replace $\mathbf{j}_m$, in Eq. (5.9), to obtain the statement of conservation of momentum

$$\frac{\partial}{\partial t}\mathbf{G} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0, \tag{5.20}$$

where

$$\mathbf{G} = \mathbf{D}\times\mathbf{B} \tag{5.21}$$

is the electromagnetic field momentum density and

$$\mathbf{T} = \mathbf{1}U - (\mathbf{ED} + \mathbf{HB}) \tag{5.22}$$

is the flux of electromagnetic field momentum density (the stress tensor).
Hints: Use the identities

$$(\boldsymbol{\nabla}\cdot\mathbf{D})\mathbf{E} + (\boldsymbol{\nabla}\times\mathbf{E})\times\mathbf{D} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{D}) + \boldsymbol{\nabla}\cdot(\mathbf{ED}), \tag{5.23a}$$

$$(\boldsymbol{\nabla}\cdot\mathbf{B})\mathbf{H} + (\boldsymbol{\nabla}\times\mathbf{H})\times\mathbf{B} = -\frac{1}{2}\boldsymbol{\nabla}(\mathbf{H}\cdot\mathbf{B}) + \boldsymbol{\nabla}\cdot(\mathbf{HB}). \tag{5.23b}$$

2. (**20 points.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$, $\varepsilon_0\mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{5.24}$$

The electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}, \qquad \mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{5.25}$$

Show that

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = aU \tag{5.26}$$

and

$$\mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = T_{ij}T_{ji} = \alpha\, U^2 + \beta\,\mathbf{G}\cdot\mathbf{S}. \tag{5.27}$$

Determine a, α, and β.
Solution: $a = 1$, $\alpha = 3$, and $\beta = -2$.

$$\mathrm{Tr}(\mathbf{T}) = T_{ii} = U \tag{5.28}$$

and

$$\mathrm{Tr}(\mathbf{T}\cdot\mathbf{T}) = T_{ij}T_{ji} = 3U^2 - 2\mathbf{G}\cdot\mathbf{S}. \tag{5.29}$$

3. (**20 points.**) The electromagnetic energy density U and the corresponding energy flux vector $\mathbf{S}$ are given by, ($\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\varepsilon_0 \mu_0 c^2 = 1$,)

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}), \qquad \mathbf{S} = \mathbf{E}\times\mathbf{H}. \tag{5.30}$$

The electromagnetic momentum density $\mathbf{G}$ and the corresponding momentum flux tensor $\mathbf{T}$ are given by

$$\mathbf{G} = \mathbf{D}\times\mathbf{B}, \qquad \mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{5.31}$$

Show that

$$\mathrm{Tr}(\mathbf{G}\times\mathbf{1}\times\mathbf{S}) = a\,(\mathbf{G}\cdot\mathbf{S}). \tag{5.32}$$

Find a. Show that

$$\mathrm{Tr}(\mathbf{G}\times\mathbf{T}\times\mathbf{S}) = 0. \tag{5.33}$$

Solution: $a = -2$.

5.3 Electromagnetic stress on a medium

See Chapter V in Volume I of Maxwell's *A Treatise on Electricity and Magnetism*. In Art 109 there, Maxwell says that these considerations were central to Faraday's investigation.

5.3.1 Electromagnetic stress on the walls of a parallel plate capacitor

1. (**20 points.**) Consider a parallel plate capacitor, with conducting plates of infinite extent separated by distance a, with uniform electric field inside the plates,

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \hat{\mathbf{z}}\dfrac{\sigma}{\varepsilon_0}, & 0 < z < a, \\ 0, & \text{otherwise}, \end{cases} \tag{5.34}$$

where σ is the charge per unit area on the plates. The magnetic field $\mathbf{B} = 0$ everywhere.

(a) Starting from the equation for conservation of electromagnetic linear momentum we have

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0. \tag{5.35}$$

Show that $\mathbf{G} = 0$. Thus, infer

$$\mathbf{f} = -\boldsymbol{\nabla}\cdot\mathbf{T}. \tag{5.36}$$

(b) Consider a cylindrical volume V with circular faces of area A parallel to the plates of capacitor. Let one face be inside the conductor on the left at $z = 0$ and the other face be inside the capacitor. Integrating over the volume V we have

$$\int_V d^3r\,\mathbf{f} = -\int_V d^3r\boldsymbol{\nabla}\cdot\mathbf{T}. \tag{5.37}$$

Using the divergence theorem we have

$$\mathbf{F} = -\oint_V d\mathbf{a}\cdot\mathbf{T} = -\oint_V da\,\hat{\mathbf{z}}\cdot\mathbf{T} \tag{5.38}$$

where we used $d\mathbf{a} = da\,\hat{\mathbf{z}}$ and recognized $\mathbf{F} = \int_V d^3r\,\mathbf{f}$ as the force on the volume V.

(c) Show that

$$\mathbf{G} = 0, \tag{5.39a}$$

$$U = \begin{cases} \dfrac{\sigma^2}{2\varepsilon_0}, & 0 < z < a, \\ 0, & \text{otherwise}, \end{cases} \tag{5.39b}$$

$$\hat{\mathbf{z}} \cdot \mathbf{T} = -U\hat{\mathbf{z}}, \tag{5.39c}$$

(d) Thus, evaluate the force per unit area on the plate at $z = 0$ of the capacitor to be

$$\frac{\mathbf{F}}{A} = U\,\hat{\mathbf{z}}. \tag{5.40}$$

5.3.2 Electromagnetic stress on parallel plate containing crossed fields

1. (**20 points.**) Let two conducting plates, with their inside faces occupying the $y = 0$ plane and $y = a$, consist of uniform positive and negative charge density σ flowing in opposite directions of $\hat{\mathbf{z}}$, respectively, described by drift velocity v_d such that the electric and magnetic field for this configuration is given by

$$\mathbf{E} = \begin{cases} \hat{\mathbf{y}}\,\dfrac{\sigma}{\varepsilon_0}, & \text{if} \quad 0 < y < a, \\ 0, & \text{otherwise}, \end{cases} \tag{5.41a}$$

$$\mathbf{B} = \begin{cases} \hat{\mathbf{z}}\,\mu_0 \sigma v_d, & \text{if} \quad 0 < y < a, \\ 0, & \text{otherwise}. \end{cases} \tag{5.41b}$$

Note,

$$cB = \beta_d E, \tag{5.42}$$

where $\beta_d = v_d/c$.

(a) Derive the expressions in Eqs. (5.41) using Gauss's law and Ampère's law, respectively.

(b) Explore the configuration in the rest frame of the flow associated with the drift velocity.

(c) Evaluate the electromagnetic stress (force per unit area) on the plate at $y = 0$. Consider the limiting case of $v_d = 0$ and match it with the results in the lecture of 2023 February 7.

(d) (This Item is not for assessment.) Discuss the relativistic transformation of the stress. Recall that $L' = L/\gamma$ and $E' = \gamma E$.

Refer the paper titled 'A simple relativistic paradox about electrostatic energy,' by W. Rindler and J. Denur, in Am. J. Phys. **56**, 795 (1988).

5.3.3 Coulomb's law as electromagnetic stress on a medium

1. (**20 points.**) Consider a charge distribution consisting of two point charges with charges equal in magnitude and opposite in sign. The positive charge $+q$ is fixed at position $\mathbf{a} = a\hat{\mathbf{z}}$ on the z axis, and the negative charge $-q$ is fixed at the origin, such that the two charges have a dipole moment $\mathbf{p} = q\mathbf{a}$. The electric field for the configuration is given by

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{(\mathbf{r} - \mathbf{a})}{|\mathbf{r} - \mathbf{a}|^3} - \frac{q}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \tag{5.43}$$

and the magnetic field $\mathbf{B} = 0$ everywhere.

(a) Starting from the equation for conservation of electromagnetic linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0, \tag{5.44}$$

infer that total force along the direction of $\hat{\mathbf{z}}$ on the charges inside volume V is given by

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \int_V d^3 r \, \mathbf{f} \cdot \hat{\mathbf{z}} = -\oint_V da \, \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}}. \tag{5.45}$$

(b) Let us choose the volume V to be a sphere of radius r centered at the negative charge. Thus, the points on the surface of this sphere satisfy

$$|\mathbf{r}| = r. \tag{5.46}$$

Let $r \ll a$. Show that on the surface of volume V

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} \hat{\mathbf{r}} + \hat{\mathbf{z}} + \mathcal{O}\left(\frac{r}{a}\right) \right]. \tag{5.47}$$

Using $\hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$ show that

$$\hat{\mathbf{r}} \cdot \mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} + \cos\theta + \mathcal{O}\left(\frac{r}{a}\right) \right], \tag{5.48a}$$

$$\mathbf{E} \cdot \hat{\mathbf{z}} = -\frac{1}{4\pi\varepsilon_0} \frac{q}{a^2} \left[\frac{a^2}{r^2} \cos\theta + 1 + \mathcal{O}\left(\frac{r}{a}\right) \right]. \tag{5.48b}$$

Verify that

$$(\mathbf{E} \cdot \mathbf{D})(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) = \frac{1}{4\pi} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^4} \left[\cos\theta \, \frac{a^4}{r^4} + 2\cos^2\theta \, \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right) \right], \tag{5.49a}$$

$$(\hat{\mathbf{r}} \cdot \mathbf{E})(\mathbf{D} \cdot \hat{\mathbf{z}}) = \frac{1}{4\pi} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^4} \left[\cos\theta \, \frac{a^4}{r^4} + \cos^2\theta \, \frac{a^2}{r^2} + \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right) \right]. \tag{5.49b}$$

(c) Show that

$$\mathbf{F} \cdot \hat{\mathbf{z}} = -\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \, r^2 \left[\frac{1}{2} (\mathbf{E} \cdot \mathbf{D})(\hat{\mathbf{r}} \cdot \hat{\mathbf{z}}) - (\hat{\mathbf{r}} \cdot \mathbf{E})(\mathbf{D} \cdot \hat{\mathbf{z}}) \right]. \tag{5.50}$$

We can complete the integral on the azimuth angle to obtain

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \lim_{r \to 0} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^2} \int_0^{\pi} \sin\theta d\theta \frac{2\pi r^2}{4\pi a^2} \left[\cos\theta \, \frac{1}{2} \frac{a^4}{r^4} + \frac{a^2}{r^2} + \mathcal{O}\left(\frac{a}{r}\right) \right]. \tag{5.51}$$

Thus, show that

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \lim_{r \to 0} \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^2} \int_0^{\pi} \sin\theta d\theta \left[\cos\theta \, \frac{1}{4} \frac{a^2}{r^2} + \frac{1}{2} \right]. \tag{5.52}$$

(d) The first equality in Eq. (5.52) is divergent in the limit $\delta \to 0$. However it goes to zero if the θ integral is completed before taking the limit. Interpret this argument. The force on the negative charge is

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^2}, \tag{5.53}$$

which is the Coulomb force.

(e) We should also be able to deduce the Coulomb law as the stress on the half-space containing one of the charges. Let us choose the volume V to be the left half space described by $z < a/2$. Then, show that

$$\mathbf{F} \cdot \hat{\mathbf{z}} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} \Big|_{z=\frac{a}{2}}. \tag{5.54}$$

Show that on the surface $z = a/2$ we have

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0} \frac{aq\,\hat{\mathbf{z}}}{\left(x^2 + y^2 + \frac{a^2}{4}\right)^{\frac{3}{2}}} \tag{5.55}$$

and

$$\hat{\mathbf{z}} \cdot \mathbf{T} \cdot \hat{\mathbf{z}} \Big|_{z=\frac{a}{2}} = -\frac{1}{2}\varepsilon_0 E^2. \tag{5.56}$$

Thus, evaluate the stress on the half space constituting volume V to be

$$\mathbf{F} \cdot \hat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{a^2}, \tag{5.57}$$

which is consistent with the interpretation of the Coulomb force.

5.3.4 Electromagnetic stress on a uniformly charged spherical ball

1. (**20 points.**) Consider a uniformly charged spherical ball of radius R with total charge q.

(a) Using Gauss's law show that the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0} \dfrac{\mathbf{r}}{R^3}, & r < R, \\ \dfrac{q}{4\pi\varepsilon_0} \dfrac{\mathbf{r}}{r^3}, & R < r. \end{cases} \tag{5.58}$$

The magnetic field $\mathbf{B} = 0$ everywhere.

(b) Starting from the equation for conservation of electromagnetic linear momentum we have

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{T} + \mathbf{f} = 0. \tag{5.59}$$

Show that $\mathbf{G} = 0$. Thus, infer

$$\mathbf{f} \cdot \hat{\mathbf{r}} = -(\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \hat{\mathbf{r}}. \tag{5.60}$$

(c) Consider a spherical volume V of radius r with the charged ball at the center. Note that $\mathbf{F} = \int_V d^3r\, \mathbf{f}$ will be zero due to spherical symmetry. To determine the electromagnetic stress (or the pressure, force per unit area,) on the sphere due to electrostatic repulsion between the constituent charges we define $F_r = \int_V d^3r\,(\mathbf{f} \cdot \hat{\mathbf{r}})$, which is the total sum of radial component of forces exerted on all the charges inside volume V by the electric and magnetic fields. Starting from

$$F_r = -\int_V d^3r (\boldsymbol{\nabla} \cdot \mathbf{T}) \cdot \hat{\mathbf{r}}, \tag{5.61}$$

after integrating by parts show that

$$F_r = -\int_V d^3r \boldsymbol{\nabla} \cdot (\mathbf{T} \cdot \hat{\mathbf{r}}) + \int_V d^3r \, \mathrm{tr}\, \mathbf{T} \cdot \boldsymbol{\nabla}\, \hat{\mathbf{r}}, \tag{5.62}$$

where we used the symmetry of $\mathbf{T}$ under transposition to write $\mathrm{tr}\, \mathbf{T} \cdot \boldsymbol{\nabla}\, \hat{\mathbf{r}} = T_{ij} \nabla_i (r_j/r)$. Then, using divergence theorem derive

$$F_r = -\oint_V da\, \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}} + \int_V d^3r \left[\frac{\mathrm{tr}(\mathbf{T}) - \hat{\mathbf{r}} \cdot \mathbf{T} \cdot \hat{\mathbf{r}}}{r}\right] \tag{5.63}$$

where we used $d\mathbf{a} = da\,\hat{\mathbf{r}}$. That is, F_r is the total radial force on the charged ball due to the flux of electromagnetic momentum across the surface enclosing volume V. Note that $F_r \neq \mathbf{F}\cdot\hat{\mathbf{r}}$, because $\mathbf{F} = 0$.

(d) Evaluate

$$U = \begin{cases} \dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^2}{R^6}, & r < R, \\ \dfrac{1}{8\pi}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{1}{r^4}, & R < r, \end{cases} \tag{5.64}$$

and

$$\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = -U. \tag{5.65}$$

Note that

$$\mathrm{tr}(\mathbf{T}) = U, \tag{5.66}$$

always.

(e) Thus, derive the expression for the radial force F_r on the charged sphere to be

$$F_r = \oint da\, U + \int_V d^3r\,\frac{2U}{r}. \tag{5.67}$$

Show that

$$\oint da\, U = \begin{cases} \dfrac{1}{2}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^4}{R^6}, & r < R, \\ \dfrac{1}{2}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{1}{r^2}, & R < r, \end{cases} \tag{5.68}$$

and

$$\int_v d^3r\,\frac{2U}{r} = \begin{cases} \dfrac{1}{4}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^4}{R^6}, & r < R, \\ \dfrac{q^2}{4\pi\varepsilon_0}\left(\dfrac{3}{4R^2} - \dfrac{1}{2r^2}\right), & R < r. \end{cases} \tag{5.69}$$

Thus, evaluate

$$F_r = \begin{cases} \dfrac{3}{4}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{r^4}{R^6}, & r < R, \\ \dfrac{3}{4}\dfrac{q^2}{4\pi\varepsilon_0}\dfrac{1}{R^2}. & R < r, \end{cases} \tag{5.70}$$

(f) Thus, calculate the electromagnetic stress, F_r/area, on the surface of the charged sphere to be

$$\frac{F_r}{4\pi R^2} = \frac{3}{16\pi}\frac{q^2}{4\pi\varepsilon_0}\frac{1}{R^4}. \tag{5.71}$$

(g) Consider the volume V to be a spherical shell of inner radius b and outer radius b', such that $R < b < b'$. Since there is no charge enclosed in the shell we expect $F_r = 0$. Verify this using Eq. (5.70). Interpret your result.

(h) Evaluate F_r for a volume constituting of a spherical shell $b < b' < R$. Is the force repulsive or attractive.

5.3.5 Electromagnetic stress on a point electric dipole

1. (**20 points.**) An electric dipole moment $\mathbf{p} = q\mathbf{a}$ consists of two equal and opposite charges q separated by a distance $\mathbf{a}$. A point electric dipole is an idealized limit of $a \to 0$, $q \to \infty$, keeping $p = aq$ fixed. The electric field of a point electric dipole moment is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\Big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\Big]. \tag{5.72}$$

The magnetic field $\mathbf{B} = 0$ everywhere. Show that the radial force F contributing to the electromagnetic stress, F/Area, on a sphere of radius r with the point dipole at the center of the sphere can be expressed in the form

$$F = \frac{c}{4\pi\varepsilon_0}\frac{p^2}{r^4}, \tag{5.73}$$

where c is a number. Find c.

2. (**20 points.**) An electric dipole moment $\mathbf{p} = q\mathbf{a}$ consists of two equal and opposite charges q separated by a distance $\mathbf{a}$. A point electric dipole is an idealized limit of $a \to 0$, $q \to \infty$, keeping $p = aq$ fixed. The electric field of a point electric dipole moment is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\Big[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\Big]. \tag{5.74}$$

The magnetic field $\mathbf{B} = 0$ everywhere.

(a) Starting from the equation for conservation of electromagnetic linear momentum we have

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0. \tag{5.75}$$

Show that $\mathbf{G} = 0$. Thus, infer

$$\mathbf{f}\cdot\hat{\mathbf{r}} = -(\boldsymbol{\nabla}\cdot\mathbf{T})\cdot\hat{\mathbf{r}}. \tag{5.76}$$

(b) Consider a spherical volume V of radius r with the charged ball at the center. Note that $\mathbf{F} = \int_V d^3r\,\mathbf{f}$ will be zero due to spherical symmetry. To determine the electromagnetic stress (or the pressure, force per unit area,) on the sphere due to electrostatic repulsion between the constituent charges we define $F_r = \int_V d^3r\,(\mathbf{f}\cdot\hat{\mathbf{r}})$, which is the total sum of radial component of forces exerted on all the charges inside volume V by the electric and magnetic fields. Starting from

$$F_r = -\int_V d^3r(\boldsymbol{\nabla}\cdot\mathbf{T})\cdot\hat{\mathbf{r}}, \tag{5.77}$$

after integrating by parts show that

$$F_r = -\int_V d^3r\boldsymbol{\nabla}\cdot(\mathbf{T}\cdot\hat{\mathbf{r}}) + \int_V d^3r\,\mathrm{tr}\,\mathbf{T}\cdot\boldsymbol{\nabla}\,\hat{\mathbf{r}}, \tag{5.78}$$

where we used the symmetry of $\mathbf{T}$ under transposition to write $\mathrm{tr}\,\mathbf{T}\cdot\boldsymbol{\nabla}\,\hat{\mathbf{r}} = T_{ij}\nabla_i(r_j/r)$. Then, using divergence theorem derive

$$F_r = -\oint_V da\,\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} + \int_V d^3r\left[\frac{\mathrm{tr}(\mathbf{T}) - \hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}}}{r}\right] \tag{5.79}$$

where we used $d\mathbf{a} = da\,\hat{\mathbf{r}}$. That is, F_r is the total radial force on the charged ball due to the flux of electromagnetic momentum across the surface enclosing volume V. Note that $F_r \neq \mathbf{F}\cdot\hat{\mathbf{r}}$, because $\mathbf{F} = 0$.

(c) Show that

$$\mathbf{E}\,\mathbf{D} = \frac{1}{4\pi}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^6}\Big[9\cos^2\theta\hat{\mathbf{r}}\hat{\mathbf{r}} - 3\cos\theta\hat{\mathbf{r}}\hat{\mathbf{z}} - 3\cos\theta\hat{\mathbf{z}}\hat{\mathbf{r}} + \hat{\mathbf{z}}\hat{\mathbf{z}}9\cos^2\theta\hat{\mathbf{r}}\hat{\mathbf{r}}\Big] \tag{5.80}$$

and

$$\hat{\mathbf{r}}\cdot\mathbf{E}\,\mathbf{D}\cdot\hat{\mathbf{r}} = \frac{1}{4\pi}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^6}4\cos^2\theta \tag{5.81}$$

and

$$U = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{1}{8\pi}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^6}\Big[1+3\cos^2\theta\Big]. \tag{5.82}$$

Then, evaluate

$$\text{tr}(\mathbf{T}) = U \tag{5.83}$$

and

$$\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = \frac{1}{8\pi}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^6}\Big[1-5\cos^2\theta\Big] \tag{5.84}$$

and

$$\text{tr}(\mathbf{T}) - \hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = \frac{1}{8\pi}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^6}8\cos^2\theta. \tag{5.85}$$

(d) Thus, evaluate

$$-\oint da\hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^4}\frac{1}{3} \tag{5.86}$$

and

$$\int_V d^3r\left[\frac{\text{tr}(\mathbf{T}) - \hat{\mathbf{r}}\cdot\mathbf{T}\cdot\hat{\mathbf{r}}}{r}\right] = \lim_{\delta\to 0}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{\delta^4}\frac{1}{3} - \frac{1}{4\pi\varepsilon_0}\frac{p^2}{r^4}\frac{1}{3}. \tag{5.87}$$

Thus, deduce the stress on a point dipole to be

$$F_r = \lim_{\delta\to 0}\frac{1}{4\pi\varepsilon_0}\frac{p^2}{\delta^4}\frac{1}{3}. \tag{5.88}$$

5.4 Electromagnetic stress on a point dyon

1. (**20 points.**) Consider the dyadic construction

$$\mathbf{K} = \mathbf{T}\times\hat{\mathbf{r}} \tag{5.89}$$

built using the vector fields,

$$\varepsilon_0\mathbf{E} = \frac{e}{4\pi}\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3}, \tag{5.90a}$$

$$\frac{\mathbf{B}}{\mu_0} = \frac{g}{4\pi}\frac{\mathbf{r}}{r^3}. \tag{5.90b}$$

5.5 Dynamic configurations

Refer: G. G. Stokes (1843), J. J. Thompson, O. Heaviside, Poincare, Lorentz, Abraham

1. (**20 points. Take home.**) Summarize Sec. III A of the article by Timothy H. Boyer titled '*Illustrations of Maxwell's term and the four conservation laws of electromagnetism*' in American Journal of Physics **87** (2019) 729. Interpret the results and answer whether the energy stored in the electromagnetic fields between the plates is increasing or decreasing with time. Verify if your answer is consistent with the direction of the flux of energy.
2. (**20 points.**) Stress on a solenoid.
3. (**20 points.**) Charged rotating shell. Will it have an angular momentum?

5.6 Plane wave configurations

1. (**20 points.**) A monochromatic plane electromagnetic wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{5.91a}$$
$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{5.91b}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are constants. Assume no charges or currents.

   (a) Using Maxwell's equations show that

$$\mathbf{k}\cdot\mathbf{E} = 0, \tag{5.92a}$$
$$\mathbf{k}\cdot\mathbf{B} = 0, \tag{5.92b}$$
$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \tag{5.92c}$$
$$\mathbf{k}\times\mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}, \tag{5.92d}$$

   where $\varepsilon_0\mu_0 = 1/c^2$.

   (b) For non-trivial cases ($\mathbf{E}_0 \neq 0$ and $\mathbf{B}_0 \neq 0$), using Eqs. (5.92), show that we have

$$ck = \omega. \tag{5.93}$$

   Then, deduce the relations

$$\mathbf{E}^*\cdot\mathbf{B} = 0, \tag{5.94}$$
$$\mathbf{E}^*\times\mathbf{B} = \hat{\mathbf{k}}\frac{1}{c}|\mathbf{E}|^2 = \hat{\mathbf{k}}c|\mathbf{B}|^2. \tag{5.95}$$

   Thus, we have

$$E = cB. \tag{5.96}$$

   (c) Evaluate the electromagnetic energy density

$$U = \frac{1}{2}\mathbf{D}^*\cdot\mathbf{E} + \frac{1}{2}\mathbf{B}^*\cdot\mathbf{H} \tag{5.97}$$

   and the electromagnetic momentum density

$$\mathbf{G} = \mathbf{D}^*\times\mathbf{B}. \tag{5.98}$$

   Then, determine the ratio U/G. What is the interpretation?

2. (**40 points.**) (Ref. Milton's lecture notes.) A plane wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{5.99}$$
$$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{5.100}$$

where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. From Maxwell's equations in free space (no charges or currents)

   (a) Determine the relation between $\mathbf{e}_0$, $\mathbf{b}_0$, and $\mathbf{k}$.

   (b) Determine the relation between ω and $\mathbf{k}$.

   (c) Verify the statement of conservation of energy for a plane wave.

   (d) Verify the statement of conservation of momentum for a plane wave.

3. (**20 points.**) The following form for the electric and magnetic field

$$\mathbf{E} = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{5.101a}$$
$$\mathbf{B} = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t+i\delta}, \tag{5.101b}$$

involes a phase difference δ between the electric and magnetic field strength. Here $\mathbf{E}_0$ and $\mathbf{B}_0$ are constants. Assume no charges or currents. Investigate if such a solution is permitted by the Maxwell equations?

4. (**60 points.**) A plane wave is incident, in vacuum, on a perfectly absorbing flat screen.

(a) Without compromising generality we can choose the screen at $z = z_a$. Starting with the statement of conservation of linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0, \tag{5.102}$$

integrate on the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Interpret the integral of force density $\mathbf{f}$ as the total force, $\mathbf{F}$, on the plate. Further, note that the integral of momentum density $\mathbf{G}$ goes to zero for infinitely small δ. Thus, obtain

$$\mathbf{F} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz\, \boldsymbol{\nabla}\cdot\mathbf{T}. \tag{5.103}$$

(b) Use divergence theorem to conclude

$$\mathbf{F} = -\oint d\mathbf{a}\cdot\mathbf{T}, \tag{5.104}$$

where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the plane wave to be incident on the side $z = z - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z + \delta$, conclude that

$$\frac{\mathbf{F}}{A} = \hat{\mathbf{z}}\cdot\mathbf{T}\big|_{z=z_a-\delta}, \tag{5.105}$$

where A is the total area of the screen. The electromagnetic stress tensor $\mathbf{T}$ in these expressions is given by

$$\mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}), \tag{5.106}$$

where U is the electromagnetic energy density,

$$U = \frac{1}{2}(\mathbf{D}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{H}). \tag{5.107}$$

(c) For the particular case when the plane wave is incident normally on the screen ($\theta = 0$ in Fig. 5.1) calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F}\cdot\hat{\mathbf{z}}}{A}. \tag{5.108}$$

Express the answer in terms of U using the properties of a plane wave: $\mathbf{k}\cdot\mathbf{E} = 0, \mathbf{k}\cdot\mathbf{B} = 0, \mathbf{E}\cdot\mathbf{B} = 0, |\mathbf{E}| = c|\mathbf{B}|$, and $kc = \omega$.

(d) Consider the case when the plane wave is incident obliquely on the screen such that $\hat{\mathbf{k}}\cdot\hat{\mathbf{z}} = \cos\theta$ and $\mathbf{H}\cdot\hat{\mathbf{z}} = 0$. Calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F}\cdot\hat{\mathbf{z}}}{A}, \tag{5.109}$$

and the force per unit area tangential to the screen by evaluating

$$\frac{\mathbf{F}\cdot\hat{\mathbf{x}}}{A}. \tag{5.110}$$

Express the answer in terms of U and θ using the properties of a plane wave.

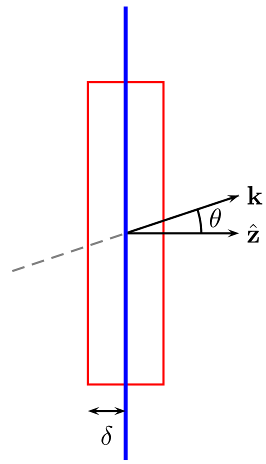

Figure 5.1: A plane wave with direction of propagation $\mathbf{k}$ incident on a screen.

5. (**60 points.**) Consider circularly polarized light of infinite extent with fields given by

$$\mathbf{E} = cB_0\big[\hat{\mathbf{x}}\cos(kz-\omega t) + \hat{\mathbf{y}}\sin(kz-\omega t)\big], \tag{5.111}$$

$$\mathbf{B} = B_0\big[-\hat{\mathbf{x}}\sin(kz-\omega t) + \hat{\mathbf{y}}\cos(kz-\omega t)\big]. \tag{5.112}$$

(a) A plane wave is characterized by $\varepsilon_0 E^2 = \mu_0 H^2$ and $\mathbf{E}\cdot\mathbf{H} = 0$. Does the above configuration satisfy the characteristics of a plane wave?

(b) Evaluate the electromagnetic energy density for the above configuration to be

$$U = \varepsilon_0 c^2 B_0^2. \tag{5.113}$$

(c) Evaluate the angular momentum density to be

$$\mathbf{L} = \mathbf{r}\times(\mathbf{D}\times\mathbf{B}) = \frac{U}{c}\,\mathbf{r}\times\hat{\mathbf{z}}. \tag{5.114}$$

(d) Determine the angular momentum flux tensor, along $\hat{\mathbf{z}}$,

$$\hat{\mathbf{z}}\cdot\mathbf{K} = -\hat{\mathbf{z}}\cdot(\mathbf{T}\times\mathbf{r}) = ?, \tag{5.115}$$

where

$$\mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{5.116}$$

(e) The above circularly polarized light is incident, in vacuum, on a perfectly absorbing flat screen. See Fig. 5.2. Without compromising generality we can choose the screen at $z = z_a$. Starting with the statement of conservation of angular momentum,

$$\frac{\partial\mathbf{L}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{K} + \mathbf{t} = 0, \tag{5.117}$$

integrate on the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Interpret the integral of torque density $\mathbf{t}$ as the total torque $\boldsymbol{\tau}$ on the plate. Further, note that the integral of angular momentum density $\mathbf{L}$ goes to zero for infinitely small δ. Thus, obtain

$$\boldsymbol{\tau} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz\, \boldsymbol{\nabla}\cdot\mathbf{K}. \tag{5.118}$$

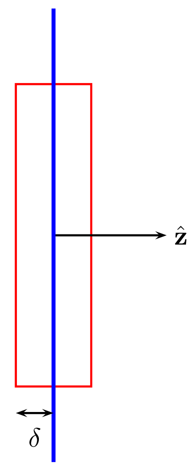

Figure 5.2: A circularly polarized light incident on a screen.

(f) Use divergence theorem to conclude

$$\boldsymbol{\tau} = -\oint d\mathbf{a} \cdot \mathbf{K}, \tag{5.119}$$

where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the circularly polarized light to be incident on the side $z = z_a - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z_a + \delta$, conclude that

$$\boldsymbol{\tau} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \hat{\mathbf{z}} \cdot \mathbf{K}\big|_{z=z_a-\delta}. \tag{5.120}$$

Use Eq. (5.115) in Eq. (5.120) and calculate the total torque on the plate.

(g) Refer [**Ohanian1986**] and problem 7.27 in Ref. [**Jackson1999**] for a complete analysis. (Will not be graded.)

Chapter 6

Macroscopic electrodynamics

6.1 Polarization

6.1.1 Effective charge density from electric polarization

1. (**Example.**) Consider a uniformly polarized slab of thickness a, that has the direction of its electric polarization (electric dipole moment density) in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{z}}\left[\theta(z) - \theta(z-a)\right] = \begin{cases} 0, & z < 0, \\ \sigma\,\hat{\mathbf{z}}, & 0 < z < a, \\ 0, & a < z, \end{cases} \tag{6.1}$$

where σ characterizes the polarization of the slab. Note that σ is dipole moment per unit volume, which has dimensions of charge per unit area.

   (a) Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{6.2}$$

   and show that

$$\rho_{\text{eff}}(\mathbf{r}) = -\sigma\delta(z) + \sigma\delta(z-a). \tag{6.3}$$

   Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density $\mathbf{P}$ leads to a surface charge density.

   (b) Find the total charge in the slab using

$$Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r}). \tag{6.4}$$

2. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{x}}\,\theta(-z), \tag{6.5}$$

where σ is the polarization per unit volume of the slab. Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P}. \tag{6.6}$$

3. (**Example.**) Consider a solid sphere of radius R with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0\,\theta(R - r), \tag{6.7}$$

where $\mathbf{P}_0$ is a uniform vector, $\theta(x)$ is the Heaviside step function, and $r^2 = x^2 + y^2 + z^2$.

(a) Show that the effective charge density due to the polarization is

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = (\mathbf{P}_0\cdot\hat{\mathbf{r}})\,\delta(r-R). \tag{6.8}$$

(b) If we choose polarization to be along the direction of $\hat{\mathbf{z}}$, that is,

$$\mathbf{P}_0 = \sigma\,\hat{\mathbf{z}}, \tag{6.9}$$

we have, using $\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = \cos\theta$,

$$\rho_{\text{eff}}(\mathbf{r}) = \sigma\cos\theta\,\delta(r-R). \tag{6.10}$$

Interpret the effective charge density as a surface charge density. Draw a diagram illustrating how the distribution of dipole moment density $\mathbf{P}$ leads to a surface charge density.

(c) Find the enclosed charge inside an arbitrary sphere of radius r using

$$Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r}) \tag{6.11}$$

for $r < R$ and $r > R$.

4. (**20 points.**) Consider a uniformly polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha\,\mathbf{r}\,\theta(R-r). \tag{6.12}$$

(a) Calculate $-\boldsymbol{\nabla}\cdot\mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R-r) + \alpha r\delta(r-R). \tag{6.13}$$

(b) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r'\,\rho_{\text{eff}}(\mathbf{r}') \tag{6.14}$$

for $r < R$ and $r > R$.

5. (**20 points.**) A permanently polarized sphere of radius R is described by the polarization vector

$$\mathbf{P}(\mathbf{r}) = \alpha r^2\,\hat{\mathbf{r}}\,\theta(R-r). \tag{6.15}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla}\cdot\mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

6. (**25 points.**) A uniformly polarized sphere of radius R is described by, $n \neq -2$,

$$\mathbf{P}(\mathbf{r}) = \alpha r^n\,\hat{\mathbf{r}}\,\theta(R-r). \tag{6.16}$$

Find the effective charge density by calculating $-\boldsymbol{\nabla}\cdot\mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R-r)$ that is interpreted as a volume charge density, and another containing $\delta(R-r)$ that can be interpreted as a surface charge density.

7. (**Example.**) Consider a solid right circular cylinder of radius R, with axis along the z axis and of infinite length, with uniform permanent polarization

$$\mathbf{P}(\mathbf{r},t) = \mathbf{P}_0\,\theta(R-\rho), \tag{6.17}$$

where $\rho^2 = x^2 + y^2$ and $\mathbf{P}_0$ is perpendicular to the axis of the cylinder.

(a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = \mathbf{P}_0\cdot\hat{\boldsymbol{\rho}}\,\delta(\rho - R). \tag{6.18}$$

(b) Discuss the case when $\mathbf{P}_0$ is parallel to the axis of the cylinder. Further, qualitatively, discuss the case if, in addition, the cylinder was of finite length in the direction of z.

8. (**20 points.**) Consider a uniformly polarized cylinder, of elliptic cross-section, described by

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\,\theta(\mu_R - \mu), \tag{6.19}$$

in terms of the elliptic coordinates (μ, ν) defined as

$$x = a\cosh\mu\cos\nu, \tag{6.20}$$

$$y = a\sinh\mu\sin\nu, \tag{6.21}$$

where $\mu \geq 0$ parameterizes confocal ellipses, $0 \leq \nu < 2\pi$ parameterizes confocal hyperbolae, such that $x = \pm a$ are the two foci of the ellipse. Thus, μ_R specifies the particular confocal ellipse. Evaluate the effective charge density

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{6.22}$$

for the polarized ellipse in terms of the elliptic coordinates and the respective unit vectors.
Hint: Unit vectors are given by the gradient of the respective coordinate surfaces. The answer to this question does not require a detailed calculation. It conceptually follows the analogous problem for spherical geometry at every step.

9. (**20 points.**) Consider a right circular cone with uniform polarization $\mathbf{P}_0$, of infinite height, apex at the origin, aperture angle $2\theta_0$, described by

$$\mathbf{P}(\mathbf{r}) = \mathbf{P}_0\,\theta_{\text{fun}}(\theta_0 - \theta), \tag{6.23}$$

where θ is the spherical polar coordinate and θ_{fun} stands for the Heaviside step function. Evaluate the effective charge density

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{6.24}$$

for the polarized cone in terms of spherical coordinates and the respective unit vectors.

6.1.2 Permanent electric polarization

1. (**Example.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization in the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{z}}\,\theta(-z), \tag{6.25}$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?

2. (**20 points.**) Consider a uniformly polarized half-slab, that occupies half of space, and has the direction of its polarization transverse to the direction $\hat{\mathbf{z}}$ that is normal to the surface of slab, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\,\hat{\mathbf{x}}\,\theta(-z), \tag{6.26}$$

where σ is the polarization of the slab. Determine the electric field, inside and outside the slab?

3. (**Example.**) Consider a solid sphere of radius R with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0\,\theta(R - r), \tag{6.27}$$

where $\mathbf{P}_0$ is a uniform vector, $\theta(x)$ is the Heaviside step function, and $r^2 = x^2 + y^2 + z^2$. We shall find the electric potential and the associated electric field inside and outside the sphere.

(a) Show that the effective charge density due to the polarization is

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} = (\mathbf{P}_0\cdot\hat{\mathbf{r}})\,\delta(r-R). \tag{6.28}$$

(b) Beginning from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{6.29}$$

show that

$$\phi(\mathbf{r}) = \frac{R^2}{4\pi\varepsilon_0}\int d\Omega\,\frac{(\mathbf{P}_0\cdot\hat{\mathbf{R}})}{|\mathbf{r}-\mathbf{R}|}. \tag{6.30}$$

Here bfr is the observation point and the integration spans the surface of the sphere, $d\Omega = \sin\theta d\theta d\phi$ and $\mathbf{R}$ is the radius vector. More explicitly we have

$$\phi(\mathbf{r}) = \frac{R^2}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi'\int_0^{\pi}\sin\theta' d\theta'\,\frac{\mathbf{P}_0\cdot(\sin\theta'\cos\phi'\hat{\mathbf{i}}+\sin\theta'\sin\phi'\hat{\mathbf{j}}+\cos\theta'\hat{\mathbf{k}})}{\sqrt{r^2+R^2-2rR\cos\gamma}}, \tag{6.31}$$

where γ is the angle between the vectors $\mathbf{r}$ and $\mathbf{R}$ and is given by

$$\cos\gamma = \sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta'. \tag{6.32}$$

(c) Out of the three vectors $\mathbf{P}_0$, $\mathbf{r}$, and $\mathbf{R}$, choose the z axis to be along $\mathbf{r}$. This renders

$$\gamma = \theta' \tag{6.33}$$

and allows for the integration to be completed using elementary substitutions. Legendre introduced the polynomials named after him primarily to evaluate these integrals without this specific choice. Complete the ϕ' integral to yield

$$\phi(\mathbf{r}) = \frac{(\mathbf{P}_0\cdot\hat{\mathbf{k}})}{4\pi\varepsilon_0}2\pi R^2\int_0^{\pi}\sin\theta' d\theta'\,\frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}}. \tag{6.34}$$

(d) Evaluate the θ' integral and show that

$$\int_0^{\pi}\sin\theta' d\theta'\,\frac{\cos\theta'}{\sqrt{r^2+R^2-2rR\cos\theta'}} = \begin{cases}\dfrac{2}{3}\dfrac{r}{R^2}, & r<R,\\[2mm] \dfrac{2}{3}\dfrac{R}{r^2}, & R<r.\end{cases} \tag{6.35}$$

(e) Thus, find the electric potential. Also, release the choice of the z axis along $\mathbf{r}$ by replacing $\mathbf{k}$ with $\mathbf{r}$. Show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{P}_0\cdot\hat{\mathbf{r}})}{r^2}\begin{cases}\dfrac{4\pi}{3}r^3, & r<R,\\[2mm] \dfrac{4\pi}{3}R^3, & R<r.\end{cases} \tag{6.36}$$

Compare this to the electric potential of a point dipole. Are they identical?

(f) Determine the electric field using

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}). \tag{6.37}$$

Show that

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\begin{cases}-\dfrac{4\pi}{3}\mathbf{P}_0, & r<R,\\[2mm] \left(\dfrac{4\pi}{3}R^3\right)\dfrac{1}{r^3}\big[3\,\hat{\mathbf{r}}\,(\mathbf{P}_0\cdot\hat{\mathbf{r}})-\mathbf{P}_0\big], & R<r.\end{cases} \tag{6.38}$$

Compare this with the electric field due to a point dipole. Plot the electric field, both outside and inside the sphere.

(g) We have the constituent relation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}. \tag{6.39}$$

Determine the expression for $\mathbf{D}$. Draw the field lines of $\mathbf{D}$, both outside and inside the sphere. How is this different from the field lines of the electric field.

4. (**20 points.**) (Based on Griffiths 4th ed., Problem 4.10.) Consider a radially polarized sphere of radius R described by

$$\mathbf{P}(\mathbf{r}) = \alpha\, \mathbf{r}\, \theta(R - r), \tag{6.40}$$

where α is constant.

(a) Calculate $-\boldsymbol{\nabla} \cdot \mathbf{P}$. Thus, find the effective charge density to be

$$\rho_{\text{eff}} = -3\alpha\theta(R - r) + \alpha r \delta(r - R). \tag{6.41}$$

(b) Using

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{6.42}$$

evaluate the electric potential to be

$$\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{2\varepsilon_0}(R^2 - r^2), & r < R, \\ 0, & R < r. \end{cases} \tag{6.43}$$

(Hint: Choose observation point $\mathbf{r}$ along $\hat{\mathbf{z}}$.)

(c) Evaulate the electric field

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \begin{cases} -\dfrac{\alpha}{\varepsilon_0}\,\mathbf{r}, & r < R, \\ 0, & r > R. \end{cases} \tag{6.44}$$

(d) Find the enclosed charge inside a sphere of radius r using

$$Q_{\text{en}} = \int d^3r'\, \rho_{\text{eff}}(\mathbf{r}') \tag{6.45}$$

for $r < R$ and $r > R$.

(e) Use Gauss's law,

$$\oint d\mathbf{a} \cdot \mathbf{E} = \frac{1}{\varepsilon_0} Q_{\text{en}}, \tag{6.46}$$

to verify the expression for the electric field in Eq. (6.44).

(f) Interpret the electric field for $r > R$ as the electric field due to the total charge inside $r \leq R$.

5. (**20 points.**) Consider a solid cylinder of radius R and infinite length with uniform permanent polarization

$$\mathbf{P}(\mathbf{r}, t) = \mathbf{P}_0\, \theta(R - \rho), \tag{6.47}$$

where $\rho^2 = x^2 + y^2$ and $\mathbf{P}_0$ is perpendicular to the axis of the cylinder. We shall find the electric potential and the electric field outside the cylinder.

(a) Show that the effective charge density is given by the expression

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P} = \mathbf{P}_0 \cdot \hat{\boldsymbol{\rho}}\, \delta(\rho - R). \tag{6.48}$$

(b) Beginning from

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \frac{\rho_{\text{eff}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{6.49}$$

after integrating by parts, and writing

$$\phi(\mathbf{r}) = -\frac{1}{4\pi\varepsilon_0}\mathbf{P}_0 \cdot \boldsymbol{\nabla} \int d^3r' \, \frac{\theta(R-\rho')}{|\mathbf{r}-\mathbf{r}'|}, \tag{6.50}$$

show that

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3r' \, \theta(R-\rho') \, \frac{\mathbf{P}_0 \cdot (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{6.51}$$

(c) Evaluate the integrals, z', ϕ', and ρ', to show that the electric potential outside the cylinder is given by

$$\phi(\mathbf{r}) = \frac{2\pi R^2}{4\pi\varepsilon_0} \, \frac{\mathbf{P_0} \cdot \boldsymbol{\rho}}{\rho^2}. \tag{6.52}$$

Hints:

i. In cylindrical coordinates we have

$$\begin{aligned} \boldsymbol{\rho} &= \rho\cos\phi\,\hat{\mathbf{i}} + \rho\sin\phi\,\hat{\mathbf{j}}, \qquad \mathbf{r} = \boldsymbol{\rho} + z\,\hat{\mathbf{k}}, && (6.53a)\\ \boldsymbol{\rho}' &= \rho'\cos\phi'\,\hat{\mathbf{i}} + \rho'\sin\phi'\,\hat{\mathbf{j}} \qquad \mathbf{r}' = \boldsymbol{\rho}' + z'\,\hat{\mathbf{k}}, && (6.53b)\\ \mathbf{P}_0 &= P_0\cos\alpha\,\hat{\mathbf{i}} + P_0\sin\alpha\,\hat{\mathbf{j}}. && (6.53c) \end{aligned}$$

Thus,

$$|\mathbf{r}-\mathbf{r}'|^2 = (z-z')^2 + |\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 \tag{6.54}$$

and

$$\mathbf{P}_0 \cdot (\mathbf{r}-\mathbf{r}') = \mathbf{P}_0 \cdot (\boldsymbol{\rho}-\boldsymbol{\rho}'). \tag{6.55}$$

ii. Complete the z' integral using

$$\int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)^{\frac{3}{2}}} = \frac{2}{a^2} \tag{6.56}$$

to obtain the result

$$\phi(\mathbf{r}) = \frac{2}{4\pi\varepsilon_0} \int d^2\rho' \, \theta(R-\rho') \, \frac{\mathbf{P}_0 \cdot (\boldsymbol{\rho}-\boldsymbol{\rho}')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}. \tag{6.57}$$

where $d^2\rho' = \rho' d\rho' d\phi'$.

iii. Choose $\phi = 0$. Then complete the ϕ' integrals using

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{1}{(1 \pm a\cos\phi)} &= \frac{1}{\sqrt{1-a^2}}, \quad |a|<1, && (6.58)\\ \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{\cos\phi}{(1 \pm a\cos\phi)} &= \pm\frac{1}{a} \mp \frac{1}{a\sqrt{1-a^2}}, \quad |a|<1, && (6.59)\\ \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{\sin\phi}{(1 \pm a\cos\phi)} &= 0, \quad |a|<1. && (6.60) \end{aligned}$$

iv. Collect (and complete cancellations) before completing the ρ' integral. The divergence iassociated to $\rho \to R$ cancels for $\rho' < R < \rho$.

(d) Evaluate the gradient of the electric potential to show that the electric field outside the cylinder is given by

$$\mathbf{E}(\mathbf{r}) = \frac{2\pi R^2}{4\pi\varepsilon_0} \, \frac{1}{\rho^2} \Big[2(\mathbf{P_0} \cdot \hat{\boldsymbol{\rho}})\hat{\boldsymbol{\rho}} - \mathbf{P}_0 \Big]. \tag{6.61}$$

6. (**20 points.**) Consider a uniformly polarized disc of radius a that has electric polarization in the radial direction, described by

$$\mathbf{P}(\mathbf{r}) = \sigma\boldsymbol{\rho}\,\theta(a-\rho)\delta(z), \tag{6.62}$$

where σ is a constant and has the dimensions of charge per unit area and $\boldsymbol{\rho}$ is the radial vector in cylindrical polar coordinates.

(a) Determine the effective charge density by evaluating

$$\rho_{\text{eff}}(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\mathbf{P} \tag{6.63}$$

and show that

$$\rho_{\text{eff}}(\mathbf{r}) = -2\sigma\theta(a-\rho)\delta(z) + \sigma a\delta(\rho-a)\delta(z). \tag{6.64}$$

Interpret the effective charge density. Find the total charge on the disc using $Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r})$.

(b) Rewrite the effective charge density in spherical polar coordinates,

$$\rho_{\text{eff}}(\mathbf{r}) = -2\sigma\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{r}\theta(a-\rho) + \sigma a\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{a}\delta(r-a). \tag{6.65}$$

Again, find the total charge on the disc using $Q_{\text{en}} = \int d^3r\,\rho_{\text{eff}}(\mathbf{r})$.

(c) Recall that the electric potential due to charged ring of radius a and total charge Q of charge density

$$\rho(\mathbf{r}') = \frac{Q}{2\pi a}\frac{\delta\left(\theta'-\frac{\pi}{2}\right)}{r'}\delta(r'-a) \tag{6.66}$$

is given by

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r. \tag{6.67}$$

Similarly, the electric potential due to a charged disc of radius a and total charge Q of charge density

$$\rho(\mathbf{r}') = \frac{Q}{\pi a^2}\frac{\delta\left(\theta'-\frac{\pi}{2}\right)}{r'}\theta(a-r'). \tag{6.68}$$

is given by

$$\phi(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r. \tag{6.69}$$

Using these results, express the electric potential due to the uniformly polarized disc in the form

$$\phi(r,\theta) = \frac{(\pi a^2)}{4\pi\varepsilon_0}\frac{\sigma}{r}\sum_{n=0}^{\infty}\alpha_n\frac{(-1)^n}{2^{2n}}\frac{(2n)!}{(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta), \quad a<r, \tag{6.70}$$

and determine α_n.
Solution: $\alpha_n = 2n/(n+1)$.

6.2 Kramers Kronig relation

1. (**Example.**) An effect in $P(t)$ caused due to $E(t)$ is described by the relation

$$P(t) = \int_{-\infty}^{\infty} dt'\,\chi(t-t')E(t'), \tag{6.71}$$

where the causal response function is characterized by

$$\chi(t-t') = \begin{cases} f(t-t'), & t' < t, \\ 0, & t < t'. \end{cases} \tag{6.72}$$

In terms of the Heaviside step function

$$\theta(t) = \begin{cases} 1, & 0 < t, \\ 0, & t < 0, \end{cases} \tag{6.73}$$

the causal response function can be expressed as

$$\chi(t) = \theta(t) f(t), \tag{6.74}$$

which by construction respects causality. The characterization of the causal response in the Fourier frequency space is the content of the Kramers Kronig relation.

(a) In terms of the Fourier transformations

$$g(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} g(\omega), \tag{6.75a}$$

$$g(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} g(t), \tag{6.75b}$$

show that

$$f(t)^* = f(t) \quad \Longrightarrow \quad f(\omega)^* = f(-\omega), \tag{6.76a}$$

$$f(-t) = -f(t) \quad \Longrightarrow \quad f(-\omega) = -f(\omega), \tag{6.76b}$$

$$\theta(t) \quad \Longrightarrow \quad \theta(\omega) = \lim_{\delta\to 0+} \frac{i}{\omega + i\delta} = \pi\delta(\omega) + i\left(\lim_{\delta\to 0} \frac{\omega}{\omega^2+\delta^2}\right). \tag{6.76c}$$

(b) We use the following characteristics of the response function:

i. The response function is real.

ii. The function $f(t)$ is left arbitrary for $t < t'$. We choose $f(t)$ to be an odd function.

iii. The response function is causal.

Using the reality and odd nature of the response, together, show that it implies that $f(\omega)$ is pure imaginary in the frequency space,

$$f(\omega) = i\,\mathrm{Im}\big[f(\omega)\big]. \tag{6.77}$$

Further, show that the imaginary part of $f(\omega)$ is odd, that is,

$$\mathrm{Im}\big[f(-\omega)\big] = -\mathrm{Im}\big[f(\omega)\big]. \tag{6.78}$$

(c) Show that the Fourier transform of the response function satisfies

$$\chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \theta(\omega-\omega') f(\omega'). \tag{6.79}$$

Thus, derive

$$\chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \Big\{ \pi\delta(\omega-\omega') + i\,\mathrm{Im}\big[\theta(\omega-\omega')\big] \Big\} i\,\mathrm{Im}\big[f(\omega)\big] \tag{6.80}$$

and deduce the content of Kramers Kronig relation,

$$\mathrm{Im}\big[\chi(\omega)\big] = \frac{1}{2}\mathrm{Im}\big[f(\omega)\big], \tag{6.81a}$$

$$\mathrm{Re}\big[\chi(\omega)\big] = -\frac{1}{\pi} \lim_{\delta\to 0} \int_{-\infty}^{\infty} d\omega' \frac{(\omega-\omega')}{(\omega-\omega')^2+\delta^2}\, \mathrm{Im}\big[\chi(\omega')\big]. \tag{6.81b}$$

(d) Show that the real part of $\chi(\omega)$ is even and the imaginary part of $\chi(\omega)$ is odd,

$$\begin{aligned} \text{Re}\big[\chi(-\omega)\big] &= -\text{Re}\big[\chi(\omega)\big], && (6.82\text{a}) \\ \text{Im}\big[\chi(-\omega)\big] &= -\text{Im}\big[\chi(\omega)\big]. && (6.82\text{b}) \end{aligned}$$

2. (**20 points.**) Evaluate the principal value of the integral, ($\delta > 0$,)

$$\int_{-\infty}^{\infty} \frac{dx}{(x + i\delta)}. \tag{6.83}$$

3. (**20 points.**) The response of a material to an electric field, in the Drude model, suitable for conductors, is described by the susceptibility function

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma}, \tag{6.84}$$

where ω_p, ω_0, and γ are material dependent parameters, and ω is the frequency of oscillation of the electric field.

(a) $[\text{Re}\chi(\omega)]$ is a measure of the square of the refractive index. Plot $[\text{Re}\chi(\omega)]$ as a function of ω.

(b) $[\text{Im}\chi(\omega)]$ is a measure of absorption of light. Plot $[\text{Im}\chi(\omega)]$ as a function of ω.

Verify that conductors absorb in a wide frequency spectrum, and display anomalous dispersion in this wide spectrum.

4. (**20 points.**) A simple model for susceptibility, suitable for insulators, is

$$\chi(\omega) = \frac{\omega_1}{\omega_0 - \omega} + i\,\pi\omega_1\delta(\omega - \omega_0), \tag{6.85}$$

where ω_0 and ω_1 represent physical parameters of a material.

(a) Note that

$$[\text{Re}\chi(\omega)] = \frac{\omega_1}{\omega_0 - \omega} \qquad \text{and} \qquad [\text{Im}\chi(\omega)] = \pi\omega_1\delta(\omega - \omega_0). \tag{6.86}$$

(b) Plot $[\text{Re}\chi(\omega)]$ and $[\text{Im}\chi(\omega)]$ with respect to ω. Verify that insulators absorb in a tiny band in the frequency spectrum, and display anomalous dispersion in this tiny band.

(c) Evaluate the right hand side of the Kramers-Kronig relation

$$[\text{Re}\chi(\omega)] = \lim_{\delta\to 0+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} [\text{Im}\chi(\omega')]\, 2\text{Re}\left\{ \frac{1}{\omega' - (\omega + i\delta)} \right\} \tag{6.87}$$

for this simple model.

5. (**20 points.**) Verify the Kramers-Kronig relation

$$[\text{Re}\chi(\omega)] = \lim_{\delta\to 0+} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} [\text{Im}\chi(\omega')]\, 2\text{Re}\left\{ \frac{1}{\omega' - (\omega + i\delta)} \right\} \tag{6.88}$$

for the dielectric model

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}. \tag{6.89}$$

6.3 Dielectric models and response functions

1. (**20 points.**) Conducting electrons, unlike bound electrons, are not confined to a particular atom. In the Drude model the motion of the conduction electrons are described by Newton's law

$$m\frac{d}{dt}\mathbf{v}(t) = -m\gamma\mathbf{v}(t) + e\mathbf{E}(t), \tag{6.90}$$

where the effects of collisions are modeled by a frictional force proportional (and opposite) to the velocity. If n_f is the (uniform) density of (free) conduction electrons, then the conduction current density is given by

$$\mathbf{J}(t) = n_f e\mathbf{v}(t). \tag{6.91}$$

(a) Solve the differential equation in Eq. (6.90) and express the solution in the form

$$\mathbf{v}(t) = \frac{e}{m}\int_{-\infty}^{t} dt'\, e^{-\gamma(t-t')}\mathbf{E}(t'). \tag{6.92}$$

Then, using Eq. (6.91) express this response in the form

$$\mathbf{J}(t) = \int_{-\infty}^{\infty} dt'\, \sigma(t-t')\varepsilon_0\mathbf{E}(t'), \tag{6.93}$$

where

$$\sigma(t) = \omega_p^2\, \theta(t)\, e^{-\gamma t} \tag{6.94}$$

and ω_p is the plasma frequency defined using

$$\omega_p^2 = \frac{n_f e^2}{m\varepsilon_0}. \tag{6.95}$$

(b) Transform the response in Eq. (6.93) into the frequency space to obtain the statement of Ohm's law

$$\mathbf{J}(\omega) = \sigma(\omega)\varepsilon_0\mathbf{E}(\omega), \tag{6.96}$$

where the conductivity $\sigma(\omega)$ is determined by the Fourier transformation

$$\sigma(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\sigma(t). \tag{6.97}$$

Complete the integration Eq. (6.97), using Eq. (6.94), to yield the Drude model for conductivity

$$\sigma(\omega) = \frac{\omega_p^2}{\gamma - i\omega}. \tag{6.98}$$

(c) For a constant electric field

$$\mathbf{E}(t) = \mathbf{E}_0 \tag{6.99}$$

evaluate the integral in Eq. (6.93), using Eq. (6.94), and show that the current density is a constant, given by

$$\mathbf{J}(t) = \frac{\omega_p^2}{\gamma}\varepsilon_0\mathbf{E}_0. \tag{6.100}$$

Use the Fourier transformation

$$\mathbf{J}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\mathbf{J}(t) \tag{6.101}$$

to deduce

$$\mathbf{J}(\omega) = \frac{\omega_p^2}{\gamma} 2\pi\delta(\omega)\varepsilon_0 \mathbf{E}_0. \tag{6.102}$$

Thus, identify the expression for static conductivity

$$\sigma(\omega) = \frac{\omega_p^2}{\gamma} 2\pi\delta(\omega) = \frac{n_f e^2}{m\varepsilon_0} \frac{1}{\gamma} 2\pi\delta(\omega). \tag{6.103}$$

The static conductivity corresponds to response at zero frequency, $\sigma(0)$.

(d) Find the current density for a pulse of infinitely short duration

$$\mathbf{E}(t) = \mathbf{e}_0 \delta(t) \tag{6.104}$$

if $\mathbf{J}(t) = 0$ for $t < 0$. Using Eq. (6.93) with Eq. (6.94) show that

$$\mathbf{J}(t) = \omega_p^2 \theta(t) e^{-\gamma t} \varepsilon_0 \mathbf{e}_0. \tag{6.105}$$

In particular, determine $\mathbf{J}(t)$ immediately after $t = 0$. Use the Fourier transformation to show that the frequency response is given by

$$\mathbf{J}(\omega) = \frac{\omega_p^2}{\gamma - i\omega} \varepsilon_0 \mathbf{e}_0. \tag{6.106}$$

2. (**20 points.**) A way of determining the sign of charge carriers in a conductor is by means of the Hall effect. A magnetic field $\mathbf{B}$ is applied perpendicular to the direction of current flow in a conductor, and as a consequence a transverse voltage drop appears across the conductor. If d is the transverse length of the conductor, and v is the average drift speed of the charge carriers, show that the voltage, in magnitude, is

$$V = vBd. \tag{6.107}$$

Estimate this potential drop (magnitude and direction) for a car driving towards North in the Northern hemisphere. How will the answer differ in the Southern hemisphere?

3. (**20 points.**) Use the statement of Ohm's law,

$$\mathbf{J} = \sigma\varepsilon_0 \mathbf{E}, \tag{6.108}$$

and generalize it for a neutral conducting fluid moving with velocity $\mathbf{v}$ as

$$\mathbf{J} = \sigma\varepsilon_0 (\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{6.109}$$

Hint: In the co-moving coordinate system, in which the fluid is at rest, show that

$$\frac{\partial}{\partial t} \to \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}, \tag{6.110}$$

$$\mathbf{E} \to \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{6.111}$$

Using Eq. (6.109) and Maxwell's equations derive

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \frac{c^2 \varepsilon_0}{\sigma} \nabla^2 \mathbf{B}. \tag{6.112}$$

(Argue that the displacement current, ignored here, gives only v^2/c^2 corrections.) For a fluid at rest this means that $\mathbf{B}$ satisfies the diffusion equation

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B}. \tag{6.113}$$

If $\mathbf{B}$ varies over a characteristic distance L, what is the characteristic time τ for the decay of the field? Estimate τ for the Earth's core, where $L \sim 10^6\,\text{m}$ and $\sigma \sim 10^7\,\text{mho/m}$. Compare this time with the current estimates for geomagnetic reversal time.

4. (**20 points.**) Plot the following as a function of ω:
   (a) Re$\chi(\omega)$ for an insulator ($\gamma \ll \omega_0$) in the Drude-Lorentz dielectric model.
   (b) Im$\chi(\omega)$ for an insulator ($\gamma \ll \omega_0$) in the Drude-Lorentz dielectric model.
   (c) Re$\chi(\omega)$ for a metal ($\omega_0 \ll \gamma$) in the Drude-Lorentz dielectric model.
   (d) Im$\chi(\omega)$ for a metal ($\omega_0 \ll \gamma$) in the Drude-Lorentz dielectric model.

   Note that the real part of the dielectric function (square of refractive index) represents dispersion. Anomalous dispersion is the behavior when the refractive index decreases with increase in frequency. Imaginary part of the dielectric function represents absorption. Observe that anomalous dispersion is accompanied by absorption? Verify that an insulator absorbs in a small band of in the electromagnetic radiation while a conductor absorbs across the whole specturm. Similarly, an insulator displays anamalous dispersion for a small band relative to conductors.

5. (**20 points.**) The charge density of a low pressure electric arc maintained using a hot filament is a called plasma. Plasma oscillations or Langmuir waves in a dilute plasma are oscillations in an electric arc described by
$$m\mathbf{a} = e\mathbf{E}(t), \tag{6.114}$$
   where we have assumed negligible friction and binding force. Using the current density
$$\mathbf{J}(\mathbf{r}, t) = n_f e\mathbf{v}(t) \tag{6.115}$$
   show that
$$\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r}, t) = \omega_p^2 \varepsilon_0 \mathbf{E}(\mathbf{r}, t). \tag{6.116}$$
   Taking the divergence in the above equation, and then using the Maxwell equation and the equation of current conservation, deduce the relation for charge density in a dilute plasma to be
$$\frac{\partial^2}{\partial t^2}\rho(\mathbf{r}, t) = -\omega_p^2 \rho(\mathbf{r}, t) \tag{6.117}$$
   whose solutions describe oscillations with angular frequency ω_p.

6. (**20 points.**) Calculate the plasma frequency of gold using
$$\omega_p^2 = \frac{n_f e^2}{m\varepsilon_0}, \tag{6.118}$$
   where n_f is the density of conduction electrons. Is this greater than or less than the frequency spectrum of visible light? Are good conductors always opaque and shiny to human eye?

7. (**40 points.**) (Refer Schwinger et al. problem 26.1 and the article in Ref. [**london1935**].)
   A simple model of a metal describes the electrons in it using Newton's law,
$$m\frac{d^2\mathbf{x}}{dt^2} + m\gamma\frac{d\mathbf{x}}{dt} + m\omega_0^2\mathbf{x} = e\mathbf{E}. \tag{6.119}$$
   Here the first term involves the acceleration of electron, ω_0-term binds the electron to the atoms, while γ-term damps the motion.

   Conductor: Conductivity in typical metals is dominated by the damping term, thus
$$m\gamma\mathbf{v} = e\mathbf{E}. \tag{6.120}$$
   The current density $\mathbf{j}$ for (constant) density n_f of conduction electrons is
$$\mathbf{j} = n_f e\mathbf{v}. \tag{6.121}$$

Using Eqs. (6.120) and (6.121) in conjunction we have Ohm's law

$$\mathbf{j} = \frac{n_f e^2}{m\gamma}\mathbf{E} = \sigma\mathbf{E}, \tag{6.122}$$

where σ is the static conductivity.

Superconductor: In 1935 Fritz London and Heinz London proposed that the current density $\mathbf{j}_s$ in a superconductor is described by the acceleration term in Eq. (6.119). That is,

$$m\frac{d\mathbf{v}}{dt} = e\mathbf{E}, \tag{6.123}$$

which together with Eq. (6.121) leads to London "acceleration equation"

$$\frac{d\mathbf{j}_s}{dt} = \frac{n_f e^2}{m}\mathbf{E}. \tag{6.124}$$

As a consequence steady currents are possible solutions when $\mathbf{E} = 0$. The insight of the London brothers led them to further propose, in addition, that the current density in a superconductor satisfies

$$\boldsymbol{\nabla}\times\left(\mathbf{j}_s + \frac{n_f e^2}{m}\mathbf{A}\right) = 0. \tag{6.125}$$

Thus, up to a freedom in the choice of gauge χ, we have the London equation

$$\mu_0\,\mathbf{j}_s = -\frac{1}{\lambda_L^2}\Big(\mathbf{A} + \boldsymbol{\nabla}\chi\Big), \tag{6.126}$$

where λ_L defined using

$$\frac{n_f e^2}{m} = \frac{1}{\lambda_L^2}\frac{1}{\mu_0} \tag{6.127}$$

is the London penetration depth which is a measure of the distance magnetic field penetrates into the surface of a superconductor. The London equation replaces Ohm's law for a superconductor. Note that the London equation is consistent with the "acceleration equation" using the gauge freedom

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi, \tag{6.128a}$$

$$\phi' = \phi - \frac{\partial\chi}{\partial t}. \tag{6.128b}$$

(a) Using London's equation show that a superconductor is characterized by the equations

$$\mu_0\frac{\partial\mathbf{j}_s}{\partial t} = \frac{1}{\lambda_L^2}\mathbf{E}, \tag{6.129}$$

$$\mu_0\boldsymbol{\nabla}\times\mathbf{j}_s = -\frac{1}{\lambda_L^2}\mathbf{B}. \tag{6.130}$$

(b) Show that the magnetic field satisfies the equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}. \tag{6.131}$$

For the static case, $\partial\mathbf{B}/\partial t = 0$, show that

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}, \tag{6.132}$$

which implies the Meissner effect, that a uniform magnetic field cannot exist inside a superconductor. In this static limit, and presuming planar geometry, it implies

$$\mathbf{B} = \mathbf{B}_0\,e^{-\frac{x}{\lambda_L}}, \tag{6.133}$$

where the interpretation of λ_L as a penetration depth is apparent. Using Eq. (6.127) calculate the penetration depth for $n_f \sim 6\times10^{28}\,/\text{m}^3$ (electron number density for gold) and show that it is of the order of tens of nanometers.

6.4 Conservation laws in macroscopic electrodynamics

6.4.1 Energy density in dispersive media

1. (**20 points.**) The constitutive relations in a nondispersive media are

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{6.134a}$$

$$\mathbf{B} = \mu \mathbf{H}, \tag{6.134b}$$

where ε and μ are constants. The ratio of speed of light in vacuum c to speed of light in the medium v is the refractive index of the medium

$$n = \frac{c}{v} = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}. \tag{6.135}$$

The theory of relativity states that velocity of energy flow can not be larger than the speed of light in vacuum. Thus, $n > 1$. Let $\mu = 1$. Consider the dielectric model

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}. \tag{6.136}$$

This is a complex number, which means a complex velocity of propagation v and a complex index of refraction

$$n = n_r + i n_i = \frac{c}{v} = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}. \tag{6.137}$$

A complex refractive index signifies that the propagation is accompanied by absorption

$$e^{-i\omega\left(t-\frac{x}{v}\right)} = e^{-i\omega\left(t-n\frac{x}{c}\right)} = e^{-n_i\frac{\omega}{c}x} e^{-i\omega\left(t-n_r\frac{x}{c}\right)}. \tag{6.138}$$

Thus, c/n_r plays the role of phase velocity and $n_i\omega/c$ is a coefficient of absorption. Plot n_r as a function of ω and verify that it crosses the line $n = 1$ near $\omega = \omega_0$. Thus, apparently, signal in a dispersive medium violates causality. This contradiction was resolved by Sommerfeld and Brillouin in 1914. Translated versions of their papers have been published in a book titled 'Wave Propagation and Group Velocity' by Brillouin in 1960. The book is available at https://archive.org. Very briefly present the resolution here.

2. (**20 points.**) Show that the energy density in a dispersive medium is given by

$$U(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi}\, e^{-i(\omega'-\omega)t}\, U(\omega,\omega'), \tag{6.139}$$

where

$$U(\omega,\omega') = \frac{1}{2}\mathbf{E}(-\omega)\cdot \frac{\left[\omega'\boldsymbol{\varepsilon}(\omega') - \omega\boldsymbol{\varepsilon}(\omega)\right]}{\omega'-\omega}\cdot \mathbf{E}(\omega') + \frac{1}{2}\mathbf{H}(-\omega)\cdot \frac{\left[\omega'\boldsymbol{\mu}(\omega') - \omega\boldsymbol{\mu}(\omega)\right]}{\omega'-\omega}\cdot \mathbf{H}(\omega'). \tag{6.140}$$

3. (**20 points.**) Show that the speed of energy flow of a monochromatic electromagnetic wave in a dispersive medium (for slowly evolving field) when both ε and μ are frequency dependent is given by

$$\frac{v_E}{c} = \left[\frac{d}{d\omega}\left(\omega\sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}\right)\right]^{-1}. \tag{6.141}$$

Determine the speed of energy flow for the case

$$\mu = \mu_0 \qquad \text{and} \qquad \frac{\varepsilon}{\varepsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \tag{6.142}$$

to be

$$\frac{v_E}{c} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1. \tag{6.143}$$

Chapter 7

Green's function

7.1 Fourier transformation

See notes on Mathematical Methods.

7.2 Method of images

1. (**20 points.**) A grounded perfectly conducting thin plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = d\,\hat{\mathbf{z}}$. Using method of images determine the electric potential at the point $\mathbf{r} = d\,\hat{\mathbf{x}} + 2d\,\hat{\mathbf{z}}$.

2. (**20 points.**) A grounded perfectly conducting thin plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = d\,\hat{\mathbf{z}}$. Using method of images determine the direction and magnitude of the electric field at the point $\mathbf{r} = d\,\hat{\mathbf{x}} + 2d\,\hat{\mathbf{z}}$.

3. (**20 points.**) A grounded perfectly conducting thin plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = a\,\hat{\mathbf{z}}$. What is the electric field at the point $\mathbf{r} = -a\,\hat{\mathbf{z}}$?

4. (**20 points.**) A grounded perfectly conducting thin plate is located at $z = 0$ plane. A positive charge q is placed at $\mathbf{r}_1 = a\,\hat{\mathbf{z}}$. A negative charge $-q$ is placed at $\mathbf{r}_2 = 2a\,\hat{\mathbf{z}}$.

   (a) Determine the magnitude and direction of the electrostatic force on the positive charge due to the negative charge.

   (b) Determine the magnitude and direction of the electrostatic force on the positive charge due to the plate. Use method of images.

   (c) Determine the magnitude and direction of the total electrostatic force on the positive charge.

5. (**20 points.**) A grounded perfect electric conductor with a planar surface occupies half of space. Two identical positive charges are placed a distance a in front of the conductor such that the distance between the two charges is $2a$. Determine the magnitude and direction of electric field at the point midway between the two charges.

6. (**20 points.**) A thin grounded perfect conductor occupies the $z = 0$ plane. A point charge q_1 is placed on one side of this conductor and another point charge q_2 is placed on the other side. The line connecting the position of the two charges is not necessarily perpendicular to the conducting plane. Let us ignore forces other than electrostatic forces in this analysis.

   (a) Identify and list the forces acting on charge q_1. Qualitatively determine the total force on charge q_1.

   (b) Identify and list the forces acting on charge q_2. Qualitatively determine the total force on charge q_2.

   (c) Identify and list the forces acting on the conductor. Qualitatively determine the total force on the conductor.

(d) Does the conductor experience a torque?

7. (**20 points.**) Consider two grounded, thin, perfect conductors occupying half planes extending radially outward from the z axis. Let these planes intersect at the z axis making an angle of 120° between them. That is, say, the two planes are $\theta = \pi/3$ and $\theta = -\pi/3$. Place a point charge on the plane $\theta = \pi/6$ as described in Figure 7.1. Determine the resulting image charge configuration, assuming that the method of images extends to these configurations analogous to optical images in a mirror.

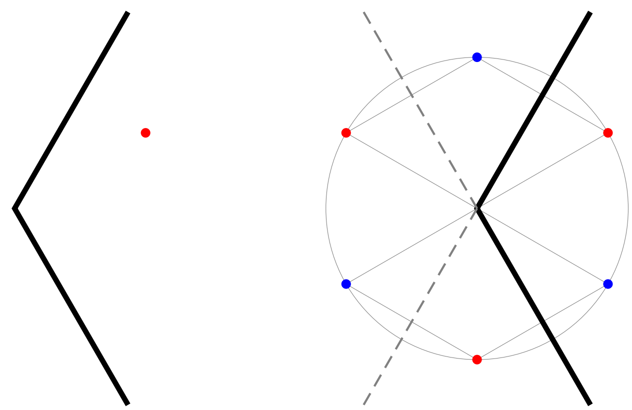

Figure 7.1: A charge near two intersecting grounded perfect conductors.

8. (**20 points.**) Consider two grounded, thin, perfect conductors occupying half planes extending radially outward from the z axis. Let these planes intersect at the z axis making an angle of 120° between them. That is, say, the two planes are $\theta = \pi/3$ and $\theta = -\pi/3$. Place a point charge on the plane $\theta = \pi/6$ as described in Figure 7.1, to the left. The resulting image charge configuration, assuming that the method of images extends to these configurations analogous to optical images in a mirror is shown in Figure 7.1, to the right.

   Let us vary the position of the point charge slightly such that it is on the plane $\theta = (\pi/6) + \varepsilon$, where $\varepsilon > 0$. Find the resulting variation in the image charge configuration.

9. (**20 points.**) A point charge q is placed near a perfectly conducting plate.

   (a) Will the charge q experience a force?

   (b) If yes, calculate the force of attraction/repulsion between the charge and conducting plate when the charge is a distance a away from the plate.

   (c) If no, why not?

10. (**Example.**) A grounded perfectly conducting thin plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = a\,\hat{\mathbf{z}}$, $a > 0$. The electric potential for this configuration is given by

$$\phi(x, y, z) = \begin{cases} 0, & z < 0, \\ \dfrac{q}{4\pi\varepsilon_0} \dfrac{1}{\sqrt{x^2 + y^2 + (z-a)^2}} - \dfrac{q}{4\pi\varepsilon_0} \dfrac{1}{\sqrt{x^2 + y^2 + (z+a)^2}}, & 0 < z. \end{cases} \tag{7.1}$$

   (a) Show that the electric potential is continuous at the surface of the conductor. That is, for an infinitely small δ,

$$\phi(x, y, 0 - \delta) = \phi(x, y, 0 + \delta). \tag{7.2}$$

(b) For a fixed $\rho = \sqrt{x^2 + y^2} \neq 0$ and $q > 0$ plot the electric potential as a function of z for $-\infty < z < \infty$. This has been plotted in Fig. 7.2 Is the force on the charge attractive or repulsive for $z > z_0$ and $z < z_0$? How much energy is required to move a test charge from a distance very far from the conducting plate to the point (ρ, z) on the surface of conductor. How does this plot change for $q < 0$?

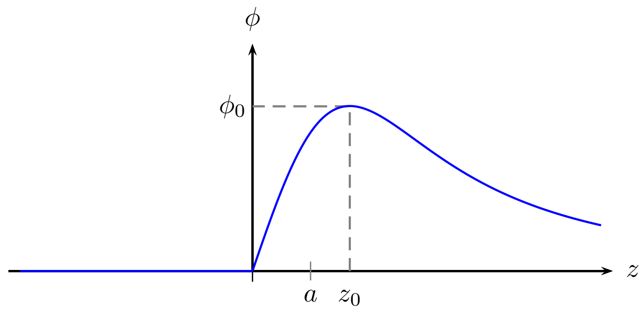

Figure 7.2: Electric potential as a function of z, for fixed ρ, for a positive charge q placed in front of a conducting plane.

11. (**20 points.**) Consider an infinite chain of equidistant alternating point charges $+q$ and $-q$ on the x-axis. Calculate the electric potential at the site of a point charge due to all other charges. This is equal to the work per point charge required to assemble such a cofiguration. In terms of the distance a between neighbouring charges we can derive an expression for this potential to be

$$V = \frac{q}{4\pi\varepsilon_0} \frac{M}{a}, \tag{7.3}$$

where M is a number defined as the Madelung constant for this hypothetical one-dimensional crystal. Determine M as an infinite sum, and evaluate the sum exactly. (Madelung contants for three-dimensional crystals involve triple sums, which are typically a challenge to evaluate because of slow convergence.)

Next, consider a point charge q placed in between two parallel grounded perfectly conducting plates. Let the plates be positioned at $z = 0$ and $z = a$. For the special situation when the charge q is equidistant from the two plates, find the pattern for the associated infinite image charges. Find the corresponding Madelung constant for this virtual crystal.

7.3 Review of Green's function

1. (**10 points.**) Verify the identity

$$\phi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \psi) - \psi \boldsymbol{\nabla} \cdot (\lambda \boldsymbol{\nabla} \phi) = \boldsymbol{\nabla} \cdot [\lambda(\phi \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \phi)], \tag{7.4}$$

which is a slight generalization of what is known as Green's second identity. Here ϕ, ψ, and λ, are position dependent functions.

2. (**10 points.**) Show that the potential for a point charge, in three spatial dimensions,

$$\phi(\mathbf{r}) = \frac{q_a}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_a|}, \tag{7.5}$$

satisfies the differential equation

$$-\varepsilon_0 \nabla^2 \phi(\mathbf{r}) = q_a \delta^{(3)}(\mathbf{r} - \mathbf{r}_a). \tag{7.6}$$

Solve the corresponding differential equation in one spatial dimension,

$$-\varepsilon_0 \frac{d^2}{dx^2}\phi(x) = q_a \delta(x - x_a). \tag{7.7}$$

Thus, apparently, the electric potential between two charges, in 1 space+1 time dimensions goes linearly with the distance r between the charges, while it goes inversely in 3 space+1 time dimensions. This leads to the conclusion that two like charges will attract and two unlike charges will repel in 1 space+1 time dimensions. This is called the Schwinger model. Read about the the Schwinger model and write a few sentences on it.
Hints:

(a) Using the definition of δ-function observe that

$$-\varepsilon_0 \frac{d^2}{dx^2}\phi(x) = 0, \quad \text{for} \quad x \neq x_a. \tag{7.8}$$

(b) Solve the homogeneous differential equation in Eq. (7.8) in terms of two integral constants in each of two regions,

$$\phi(x) = \begin{cases} a_1 x + b_1, & x < x_a, \\ a_2 x + b_2, & x > x_a. \end{cases} \tag{7.9}$$

(c) Integrate Eq. (7.7) from $x = x_a - \delta$ to $x = x_a + \delta$, for infinitesimal $\delta > 0$, to derive the boundary condition on

$$\frac{d}{dx}\phi(x). \tag{7.10}$$

(d) Argue that, for consistency, we also require the boundary condition

$$\phi(x_a - \delta) = \phi(x_a + \delta). \tag{7.11}$$

(e) Use the boundary conditions to determine two of the four integral constants in Eq. (7.9). In particular find $a_2 - a_1$ and $b_2 - b_1$. The solutions can be expressed in the form

$$\phi(x) = -\frac{q}{2\varepsilon_0}|x - x_a| + ax + b, \tag{7.12}$$

where $2a = a_1 + a_2$ and $2b = b_1 + b_2$.

3. (**10 points.**) Show that

$$\bar{\delta}(x) = -x\frac{d}{dx}\delta(x) \tag{7.13}$$

is also a model for the δ-function by showing that

$$\int_{-\infty}^{\infty} dx\, \bar{\delta}(x) f(x) = f(0). \tag{7.14}$$

Hint: Integrate by parts.

4. (**40 points.**) Verify that

$$\frac{d}{dz}|z| = \theta(z) - \theta(-z), \tag{7.15}$$

where $\theta(z) = 1$, if $z > 0$, and 0, if $z < 0$. Further, verify that

$$\frac{d^2}{dz^2}|z| = 2\,\delta(z). \tag{7.16}$$

Also, argue that, for a well defined function $f(z)$, the replacement

$$f(z)\delta(z) = f(0)\delta(z) \tag{7.17}$$

is justified. Using Eq. (7.15), Eq. (7.16), and Eq. (7.17), verify (by substituting the solution into the differential equation) that

$$g(z) = \frac{1}{2k}\, e^{-k|z|} \tag{7.18}$$

is a particular solution of the differential equation

$$\left(-\frac{d^2}{dz^2} + k^2\right) g(z) = \delta(z). \tag{7.19}$$

5. (**70 points.**) A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) x(t) = F(t), \tag{7.20}$$

where ω is the angular frequency of the oscillator and $F(t)$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) G(t,t') = \delta(t-t'). \tag{7.21}$$

The continuity conditions satisfied by the Green function are

$$\frac{d}{dt} G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta} = -1 \tag{7.22}$$

and

$$G(t,t')\Big|_{t=t'-\delta}^{t=t'+\delta} = 0. \tag{7.23}$$

(a) Verify that a particular solution,

$$G_R(t-t') = -\frac{1}{\omega}\theta(t-t')\sin\omega(t-t'), \tag{7.24}$$

which is referred to as the retarded Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 3 and $\lim_{x\to\infty} \sin x/x = 0$.

(b) Verify that another particular solution,

$$G_A(t-t') = \frac{1}{\omega}\theta(t'-t)\sin\omega(t-t'), \tag{7.25}$$

which is referred to as the advanced Green's function, satisfies the Green function differential equation and the continuity conditions.
Hint: Use problem 3 and $\lim_{x\to\infty} \sin x/x = 0$.

(c) Show that the difference of the two particular solutions above,

$$G_R(t-t') - G_A(t-t'), \tag{7.26}$$

satisfies the homogeneous differential equations

$$-\left(\frac{d^2}{dt^2} + \omega^2\right) G_0(t,t') = 0. \tag{7.27}$$

6. (**20 points.**) A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t)=A(t), \tag{7.28}$$

where ω is the angular frequency of the oscillator and $A(t)=F(t)/m$ is the forcing function. The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{7.29}$$

The Wronskian $G(t,t')$ and $x(t)$ is

$$W[G(t,t'),x(t')]=G(t,t')\frac{d}{dt'}x(t')-x(t')\frac{d}{dt'}G(t,t'). \tag{7.30}$$

Show that

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)W[G(t,t'),x(t')]=W[\delta(t-t'),x(t')]. \tag{7.31}$$

Then, evaluate

$$\int_{-\infty}^{+\infty}dt'\frac{d}{dt'}W[\delta(t-t'),x(t')]. \tag{7.32}$$

7. (**30 points.**) A forced harmonic oscillator is described by the differential equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x(t)=A(t), \tag{7.33}$$

with appropriate initial conditions, say,

$$x(0)=-A_0, \qquad \text{and} \qquad \dot{x}(0)=\left.\frac{dx(t)}{dt}\right|_{t=0}=0. \tag{7.34}$$

Here ω is the angular frequency of the oscillator and $A(t)=F(t)/m$ is a priori given forcing function (or the source). The corresponding Green's function satisfies

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t,t')=\delta(t-t'). \tag{7.35}$$

(a) Show that the solution, $x(t)$, to the differential equation in Eq. (7.33), is given in terms of the Green function by

$$x(t) = \int_{-\infty}^{+\infty}dt'G(t,t')F(t')+\int_{-\infty}^{+\infty}dt'\frac{d}{dt'}\left[\left(\dot{x}(t')-x(t')\frac{d}{dt'}\right)G(t,t')\right] \tag{7.36}$$

$$= \int_{-\infty}^{+\infty}dt'G(t,t')F(t') + \lim_{\tau_2\to+\infty}\left[\dot{x}(\tau_2)-x(\tau_2)\frac{d}{d\tau_2}\right]G(t,\tau_2)-\lim_{\tau_1\to-\infty}\left[\dot{x}(\tau_1)-x(\tau_1)\frac{d}{d\tau_1}\right]G(t,\tau_1), \tag{7.37}$$

where the limiting variables in the second equality are constructed such that $\tau_1<\{t,t'\}<\tau_2$.

(b) The corresponding homogeneous differential equation is

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)x_0(t)=0. \qquad \text{and} \qquad -\left(\frac{d^2}{dt^2}+\omega^2\right)G_0(t,t')=0. \tag{7.38}$$

i. Show that for a Greens function, $G(t,t')$, that solves Eq. (7.35),

$$G(t,t') + G_0(t,t') \tag{7.39}$$

is also a solution to Eq. (7.35).

ii. Show that the homogeneous solution of the Greens function does not contribute to $x(t)$ by showing that

$$\int_{-\infty}^{+\infty} dt' G_0(t,t') F(t') + \int_{-\infty}^{+\infty} dt' \frac{d}{dt'} \left[\left(\dot{x}(t') - x(t') \frac{d}{dt'} \right) G_0(t,t') \right] = 0. \tag{7.40}$$

iii. Argue that the surface terms in Eq. (7.35) satisfy the homogeneous differential equation in Eq. (7.38)

$$-\left(\frac{d^2}{dt^2} + \omega^2 \right) \left[\left(\dot{x}(\tau) - x(\tau) \frac{d}{d\tau} \right) \bar{G}(t,\tau) \right] = 0, \tag{7.41}$$

because the surface points, denoted by τ above, never equals the variable t, i.e. $\tau \neq t$.

(c) Beginning with Eq. (7.35) derive the continuity conditions satisfied by the Green function at $t = t'$ to be

$$\frac{d}{dt} G(t,t') \Big|_{t=t'-\delta}^{t=t'+\delta} = -1 \tag{7.42}$$

and

$$G(t,t') \Big|_{t=t'-\delta}^{t=t'+\delta} = 0. \tag{7.43}$$

(d) For all points, except $t = t'$, the differential Eq. (7.35) has no source term and thus reads like the equation for $G_0(t,t')$ in Eq.(7.38). This equation has oscillatory solutions, which could have different behavior at $t < t'$ and $t > t'$, except for the constraint imposed by the continuity conditions in Eqs. (7.42) and (7.43). In terms of four arbitrary functions of t', A, B, C, and D, we can write

$$G(t,t') = \begin{cases} A(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t}, & \text{if } t < t', \\ C(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t}, & \text{if } t > t'. \end{cases} \tag{7.44}$$

Imposing the continuity conditions in Eqs. (7.42) and (7.43) derive the following equations constraining $A(t')$, $B(t')$, $C(t')$, and $D(t')$:

$$[C(t') - A(t')]\, e^{i\omega t'} + [D(t') - B(t')]\, e^{-i\omega t'} = 0, \tag{7.45}$$

$$[C(t') - A(t')]\, e^{i\omega t'} - [D(t') - B(t')]\, e^{-i\omega t'} = \frac{i}{\omega}. \tag{7.46}$$

(e) Using the continuity conditions and without imposing any boundary conditions solve for $G(t,t')$ in the following four forms:

$$G(t,t') = A(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t} + G_R(t-t') \tag{7.47a}$$

$$= C(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t} + G_A(t-t') \tag{7.47b}$$

$$= A(t')\, e^{i\omega t} + D(t')\, e^{-i\omega t} + G_F(t-t') \tag{7.47c}$$

$$= C(t')\, e^{i\omega t} + B(t')\, e^{-i\omega t} + G_W(t-t') \tag{7.47d}$$

where

$$G_R(t-t') = -\frac{1}{\omega}\, \theta(t-t')\, \sin\omega(t-t'), \tag{7.48a}$$

$$G_A(t-t') = +\frac{1}{\omega}\, \theta(t'-t)\, \sin\omega(t-t'), \tag{7.48b}$$

$$G_F(t-t') = -\frac{1}{\omega}\frac{1}{2i} \left[+\theta(t-t')\, e^{i\omega(t-t')} + \theta(t'-t)\, e^{-i\omega(t-t')} \right], \tag{7.48c}$$

$$G_W(t-t') = -\frac{1}{\omega}\frac{1}{2i} \left[-\theta(t'-t)\, e^{i\omega(t-t')} - \theta(t-t')\, e^{-i\omega(t-t')} \right], \tag{7.48d}$$

and the subscripts stand for retarded, advanced, Feynman, and Wheeler, respectively. Recognize that the above four forms are special cases of the following general expression

$$-\frac{1}{\omega}\frac{1}{2i}\left[a\,\theta(t-t')-b\,\theta(t'-t)\right]e^{i\omega(t-t')}+\frac{1}{\omega}\frac{1}{2i}\left[c\,\theta(t-t')-d\,\theta(t'-t)\right]e^{-i\omega(t-t')}, \tag{7.49}$$

where the numerical constants a, b, c, and d, are arbitrary to the extent that they obey the constraints $a+b=1$, and $c+d=1$. The special cases, $a=1$, $c=1$, corresponds to G_R; $a=0$, $c=0$, corresponds to G_A; $a=1$, $c=0$, corresponds to G_F; and $a=0$, $c=1$, corresponds to G_W, respectively.

(f) Show that we can write

$$x(t)=\int_{-\infty}^{+\infty}dt'G(t-t')F(t')+\alpha_0\,e^{i\omega t}+\beta_0\,e^{-i\omega t}, \tag{7.50}$$

where α_0 and β_0 are the arbitrary numerical constants. Use the initial conditions of Eq. (7.34) in Eq. (7.50), in conjunction with Eq. (7.49), to derive

$$\alpha_0 = -\frac{A}{2}+a\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0}dt'F(t')e^{-i\omega t'}-b\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty}dt'F(t')e^{-i\omega t'}, \tag{7.51a}$$

$$\beta_0 = -\frac{A}{2}-c\,\frac{1}{\omega}\frac{1}{2i}\int_{-\infty}^{0}dt'F(t')e^{+i\omega t'}+d\,\frac{1}{\omega}\frac{1}{2i}\int_{0}^{+\infty}dt'F(t')e^{+i\omega t'}. \tag{7.51b}$$

Using the above expressions for α_0 and β_0 in Eq. (7.50) obtain

$$x(t)=-A\cos\omega t-\frac{1}{\omega}\int_0^t dt'F(t')\sin\omega(t-t'), \tag{7.52}$$

which uses $a+b=1$ and $c+d=1$.

8. (**30 points.**) Consider the Green function equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t)=\delta(t). \tag{7.53}$$

Verify, by substituting into Eq. (7.53), that

$$G(t)=-\frac{1}{\omega}\theta(t)\sin\omega(t), \tag{7.54}$$

is a particular solution to the Green's function equation.

9. (**25 points.**) The electrostatic electric potential, $\phi(\mathbf{r})$, for a unit point charge placed at the origin satisfies

$$-\nabla^2\phi(\mathbf{r})=\delta^{(3)}(\mathbf{r}). \tag{7.55}$$

Verify, by substituting into Eq. (7.55), that

$$\phi(\mathbf{r})=\frac{1}{4\pi r} \tag{7.56}$$

is a particular solution for $\phi(\mathbf{r})$.
Hint: Verify that the left hand side of Eq. (7.55) satisfies the properties of δ-function in three dimensions, i.e., it is zero for $\mathbf{r}\neq 0$ and the integral over a volume including $\mathbf{r}=0$ is 1.

10. (**50 points.**) The electric potential is given in terms of the Greens function by the expression

$$\phi(\mathbf{r})=\frac{1}{\varepsilon_0}\int d^3r'G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'). \tag{7.57}$$

A representation of the Greens function that is suitable for the case when the charge density is a function of z alone is

$$G(\mathbf{r},\mathbf{r}')=\int\frac{d^2k_\perp}{(2\pi)^2}\,e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp}\,\frac{1}{2k_\perp}\,e^{-k_\perp|z-z'|}. \tag{7.58}$$

(a) Express the electric potential due to an infinitely thin plate described by the charge density $\rho(\mathbf{r}) = \sigma\delta(z-a)$ in the form

$$\phi(\mathbf{r}) = \frac{\sigma}{2\varepsilon_0} \lim_{k_\perp \to 0} \left[\frac{1}{k_\perp} - |z-a| \right]. \tag{7.59}$$

Hint: Start by evaluating the z' integral, that involves a δ-function integral, after substituting the expressions for $G(\mathbf{r}, \mathbf{r}')$ and $\rho(\mathbf{r}')$ into Eq. (7.57). Use the δ-function representation,

$$\int_{-\infty}^{\infty} dx\, e^{ikx} = 2\pi\delta(k), \tag{7.60}$$

to complete the integrations on x' and y'. Then complete the k_x and k_y integral in the form of limits after expanding the exponential using Taylor series.

(b) Show that the electric field then is

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \begin{cases} \dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, & z > a, \\ -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, & z < a, \end{cases} \tag{7.61}$$

11. (**40 points.**) Consider the integral equation

$$K(t', t'') + i \int_0^t d\tau \Big[1 + i t_<(t', \tau) \Big] K(\tau, t'') = \delta(t' - t''), \qquad 0 \le \{t', t''\} \le t, \tag{7.62}$$

where $t_<(t', \tau)$ stands for minimum of t' and τ.

(a) By differentiating the above integral equation in Eq. (7.62) twice with respect to t' obtain the differential equation satisfied by $K(t', t'')$:

$$\left[\frac{\partial^2}{\partial t'^2} + 1 \right] K(t', t'') = \frac{\partial^2}{\partial t'^2} \delta(t' - t''). \tag{7.63}$$

(b) Deduce the boundary conditions on $K(t', t'')$ from Eq. (7.62):

$$K(0, t'') = -i \int_0^t d\tau K(\tau, t''), \tag{7.64a}$$

$$K(t, t'') = K(0, t'') + \int_0^t d\tau\, \tau K(\tau, t''). \tag{7.64b}$$

Hint: Presume that the δ-function in Eq. (7.62) does not contribute at $t' = 0$ and $t' = t$. This assumption does not effect the solution, but leads to non-trivial contributions at the boundaries of integrals involving $K(t, t'')$.

(c) In terms of a Green's function $M(t', t'')$, which satisfies

$$\left[\frac{\partial^2}{\partial t'^2} + 1 \right] M(t', t'') = \delta(t' - t''), \tag{7.65}$$

write

$$K(t', t'') = \frac{\partial^2}{\partial t'^2} M(t', t'') = \delta(t' - t'') - M(t', t''). \tag{7.66}$$

(d) Derive the continuity conditions for $M(t',t'')$, which are dictated by Eq. (7.65), to be

$$\{M(t',t'')\}_{t'=t''+\delta} - \{M(t',t'')\}_{t'=t''-\delta} = 0, \tag{7.67a}$$

$$\left\{\frac{\partial}{\partial t'}M(t',t'')\right\}_{t'=t''+\delta} - \left\{\frac{\partial}{\partial t'}M(t',t'')\right\}_{t'=t''-\delta} = 1, \tag{7.67b}$$

Additionally, the boundary conditions on $M(t',t'')$ are prescribed by the boundary conditions on $K(t',t'')$ in Eqs. (7.64a) and (7.64b).

(e) Write the solution to $M(t',t'')$ in the form

$$M(t',t'') = \begin{cases} \alpha(t'')\sin t' + \beta(t'')\cos t', & 0 \le t' < t'' \le t, \\ \eta(t'')\sin t' + \xi(t'')\cos t', & 0 \le t'' < t' \le t, \end{cases} \tag{7.68}$$

in terms of four arbitrary constants. Use the continuity conditions (7.67) to determine two of the four constants to obtain

$$K(t',t'') = \delta(t'-t'') - \alpha(t'')\sin t' - \xi(t'')\cos t' - \sin t_> \cos t_<, \tag{7.69}$$

where we have suppressed the t' and t'' dependence in $t_<(t',t'')$ and $t_>(t',t'')$.

(f) Use the expression for $K(t',t'')$ in Eq. (7.69) into Eqs. (7.64a) and (7.64b) to obtain the equations determining $\alpha(t'')$ and $\xi(t'')$ to be

$$\alpha(t'')i[1-\cos t] + \xi(t'')[1+i\sin t] \;=\; i\cos t\cos t'' - \sin t'', \tag{7.70a}$$

$$\alpha(t'')\cos t - \xi(t'')\sin t \;=\; -\cos t\cos t'', \tag{7.70b}$$

and further obtain

$$\alpha(t'') \;=\; -e^{-it}\cos(t-t''), \tag{7.71a}$$

$$\xi(t'') \;=\; ie^{-i(t-t'')}\cos t. \tag{7.71b}$$

(g) Using Eqs. (7.71a) and (7.71b) in Eq. (7.69) obtain the solution to $K(t',t'')$ in the form

$$K(t',t'') = \delta(t'-t'') - i\cos(t-t')\cos(t-t'') - \sin(t-t_<)\cos(t-t_>). \tag{7.72}$$

(h) By substitution verify that Eq. (7.72) satisfies the original integral equation (7.62).

7.4 Planar geometry: Method of images for dielectrics

1. (**40 points.**) The expression for the electric potential due to a point charge placed in front of a perfectly conducting semi-infinite slab, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z, \end{cases} \tag{7.73}$$

is given in terms of the reduced Green function that satisfies the differential equation $(0 < \{z,z'\})$

$$-\left[\frac{\partial^2}{\partial z^2} - k^2\right]\varepsilon_0 g(z,z') = \delta(z-z') \tag{7.74}$$

with boundary conditions requiring the reduced Green's function to vanish at $z=0$ and at $z\to\infty$.

(a) Construct the reduced Green function in the form

$$\varepsilon_0 g(z,z') = \begin{cases} Ae^{kz} + Be^{-kz}, & 0 < z < z', \\ Ce^{kz} + De^{-kz}, & 0 < z' < z, \end{cases} \tag{7.75}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\varepsilon_0 g(0,z') = 0, \tag{7.76a}$$
$$\varepsilon_0 g(\infty,z') = 0, \tag{7.76b}$$
$$\varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{7.76c}$$
$$\partial_z \varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{7.76d}$$

(b) Express the solution in the form

$$\varepsilon_0 g(z,z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{7.77}$$

(c) Deduce the method of images from the above solution.

2. (**40 points.**) The expression for the electric potential due to a point charge placed in between two parallel grounded perfectly conducting semi-infinite slabs, described by

$$\frac{\varepsilon(z)}{\varepsilon_0} = \begin{cases} \infty, & z < 0, \\ 1, & 0 < z < a, \\ \infty, & a < z, \end{cases} \tag{7.78}$$

is given in terms of the reduced Green function that satisfies the differential equation ($0 < \{z, z'\} < a$)

$$\left[-\frac{\partial^2}{\partial z^2} + k^2 \right] \varepsilon_0 g(z,z') = \delta(z - z') \tag{7.79}$$

with boundary conditions requiring the reduced Green's function to vanish at $z = 0$ and $z = a$.

(a) Construct the reduced Green's function in the form

$$\varepsilon_0 g(z,z') = \begin{cases} A \sinh kz + B \cosh kz, & 0 < z < z' < a, \\ C \sinh kz + D \cosh kz, & 0 < z' < z < a, \end{cases} \tag{7.80}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$\varepsilon_0 g(0,z') = 0, \tag{7.81a}$$
$$\varepsilon_0 g(a,z') = 0, \tag{7.81b}$$
$$\varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{7.81c}$$
$$\partial_z \varepsilon_0 g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{7.81d}$$

(b) After using conditions in Eqs. (7.81a) and (7.81b) show that the reduced Green's function can be expressed in the form

$$\varepsilon_0 g(z,z') = \begin{cases} A \sinh kz, & 0 < z < z' < a, \\ C' \sinh k(a - z), & 0 < z' < z < a, \end{cases} \tag{7.82}$$

where $C' = -C/\cosh ka$. Then, use Eqs. (7.81c) and (7.81d) to show that

$$\varepsilon_0 g(z,z') = \begin{cases} \dfrac{\sinh kz \sinh k(a-z')}{k \sinh ka}, & 0 < z < z' < a, \\ \dfrac{\sinh kz' \sinh k(a-z)}{k \sinh ka}, & 0 < z' < z < a. \end{cases} \tag{7.83}$$

(c) Take the limit $ka \to \infty$ in your solution above, (which corresponds to moving the slab at $z = a$ to infinity,) to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka\to\infty} \varepsilon_0 g(z,z') = \frac{1}{2k} e^{-k|z-z'|} - \frac{1}{2k} e^{-k|z|} e^{-k|z'|}. \tag{7.84}$$

This should serve as a check for your solution to the reduced Green's function. Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{7.85}$$

3. (**40 points.**) Consider the differential equation

$$\left[-\frac{\partial}{\partial z} \varepsilon(z) \frac{\partial}{\partial z} + \varepsilon(z) k_\perp^2 \right] g_\varepsilon(z,z') = \delta(z-z'), \tag{7.86}$$

for the case

$$\varepsilon(z) = \begin{cases} \varepsilon_2, & z < 0, \\ \varepsilon_1, & 0 < z, \end{cases} \tag{7.87}$$

satisfying the boundary conditions

$$g_\varepsilon(-\infty, z') = 0, \tag{7.88a}$$
$$g_\varepsilon(+\infty, z') = 0. \tag{7.88b}$$

(a) Verify, by integrating Eq. (7.86) around $z = z'$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z,z') \Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{7.89a}$$
$$\varepsilon(z) \frac{\partial}{\partial z} g_\varepsilon(z,z') \Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{7.89b}$$

(b) Verify, by integrating Eq. (7.86) around $z = 0$, that the Green function satisfies the continuity conditions

$$g_\varepsilon(z,z') \Big|_{z=0-\delta}^{z=0+\delta} = 0, \tag{7.90a}$$
$$\varepsilon(z) \frac{\partial}{\partial z} g_\varepsilon(z,z') \Big|_{z=0-\delta}^{z=0+\delta} = 0. \tag{7.90b}$$

(c) For $z' < 0$, construct the solution in the form

$$g_\varepsilon(z,z') = \begin{cases} A_1 e^{k_\perp z} + B_1 e^{-k_\perp z}, & z < z' < 0, \\ C_1 e^{k_\perp z} + D_1 e^{-k_\perp z}, & z' < z < 0, \\ E_1 e^{k_\perp z} + F_1 e^{-k_\perp z}, & z' < 0 < z. \end{cases} \tag{7.91}$$

Determine the constants using the boundary conditions and continuity conditions.

(d) For $0 < z'$, construct the solution in the form

$$g_\varepsilon(z,z') = \begin{cases} A_2 e^{k_\perp z} + B_2 e^{-k_\perp z}, & z < 0 < z', \\ C_2 e^{k_\perp z} + D_2 e^{-k_\perp z}, & 0 < z < z', \\ E_2 e^{k_\perp z} + F_2 e^{-k_\perp z}, & 0 < z' < z. \end{cases} \tag{7.92}$$

Determine the constants using the boundary conditions and continuity conditions.

(e) Thus, find the solution

$$g_\varepsilon(z,z') = \begin{cases} \frac{1}{\varepsilon_2}\frac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \frac{1}{\varepsilon_2}\frac{1}{2k_\perp}\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) e^{-k_\perp|z|}e^{-k_\perp|z'|}, & z' < 0, \\ \frac{1}{\varepsilon_1}\frac{1}{2k_\perp} e^{-k_\perp|z-z'|} + \frac{1}{\varepsilon_1}\frac{1}{2k_\perp}\left(\frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) e^{-k_\perp|z|}e^{-k_\perp|z'|}, & 0 < z'. \end{cases} \tag{7.93}$$

4. (**50 points.**) Let the space be filled with two dielectric materials, with a discontinuity at $z = 0$, such that

$$\varepsilon(\mathbf{r}) = \varepsilon_2\theta(-z) + \varepsilon_1\theta(z), \tag{7.94}$$

where

$$\varepsilon_2 < \varepsilon_1. \tag{7.95}$$

In addition there is a point charge at q at

$$\mathbf{r}' = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + a\,\hat{\mathbf{z}}. \tag{7.96}$$

In the following we shall determine the electric potential and electric field everywhere for this configuration.

(a) Starting from the Maxwell equations (in vacuum) the electric potential for a single point charge at $\mathbf{r}'$ is

$$-\varepsilon_0\nabla^2\phi(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{7.97}$$

Construct the corresponding Green's function to satisfy

$$-\varepsilon_0\nabla^2 G_0(\mathbf{r}-\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'), \tag{7.98}$$

which is obtained by replacing the charge density with that of a point charge of unit magnitude that is achieved by simply choosing $q = 1$ in this case. The electric potential is in general given in terms of the Green's function by the superposition principle

$$\phi(\mathbf{r}) = \int d^3r' G_0(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}'), \tag{7.99}$$

which for a point charge in vacuum simply reads

$$\phi(\mathbf{r}) = q\,G_0(\mathbf{r},\mathbf{r}'), \qquad G_0(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}, \tag{7.100}$$

Observe that the configuration under consideration, a point charge in a planar dielectric region, has translation symmetry in x and y directions. Thus, we use Fourier transformation in these coordinates to write

$$G_0(\mathbf{r},\mathbf{r}') = \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp} g_0(z,z';k_\perp), \tag{7.101}$$

where $g_0(z,z';k_\perp)$ is the Fourier transform of $G_0(\mathbf{r},\mathbf{r}')$ in the x and y coordinates. Here the subsript $\perp$ is the projection in the plane perpendicular to $\hat{\mathbf{z}}$. Show that the reduced Green's function $g_0(z,z';k_\perp)$ satisfies the differential equation

$$-\left(\frac{d^2}{dz^2} - k_\perp^2\right)\varepsilon_0 g_0(z,z';k_\perp) = \delta(z-z'). \tag{7.102}$$

Show that

$$g_0(z,z';k_\perp) = \frac{1}{\varepsilon_0}\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}. \tag{7.103}$$

Thus, find the identity

$$\begin{aligned}\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} \\ &= \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp}\frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}.\end{aligned} \tag{7.104}$$

(b) Starting from the macroscopic Maxwell equations the electric potential for a single point charge at $\mathbf{r}'$ in the presence of a dielectric material is

$$-\boldsymbol{\nabla}\cdot\Big[\varepsilon(\mathbf{r})\boldsymbol{\nabla}\Big]\phi(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{7.105}$$

Construct the corresponding Green's function to satisfy

$$-\boldsymbol{\nabla}\cdot\Big[\varepsilon(\mathbf{r})\boldsymbol{\nabla}\Big]G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{7.106}$$

Show that the corresponding reduced Green's function $g(z,z';k_\perp)$ satisfies the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}+\varepsilon(z)k_\perp^2\right]g(z,z';k_\perp) = \delta(z-z'), \tag{7.107}$$

where

$$\varepsilon(z) = \begin{cases}\varepsilon_2, & z<0,\\ \varepsilon_1, & 0<z.\end{cases} \tag{7.108}$$

Show that

$$g(z,z') = \begin{cases}\dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp}e^{-k_\perp|z-z'|} + \dfrac{1}{\varepsilon_2}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right)e^{-k_\perp|z|}e^{-k_\perp|z'|}, & z'<0,\\[2ex] \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp}e^{-k_\perp|z-z'|} + \dfrac{1}{\varepsilon_1}\dfrac{1}{2k_\perp}\left(\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right)e^{-k_\perp|z|}e^{-k_\perp|z'|}, & 0<z'.\end{cases} \tag{7.109}$$

(c) Use the identity in Eq. (7.104) to show that

$$\phi(\mathbf{r}) = q\,G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\varepsilon_1}\frac{q}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon_1}\frac{q_{\text{im}}}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|}, \tag{7.110}$$

where

$$q_{\text{im}} = -q\left(\frac{\varepsilon_2-\varepsilon_1}{\varepsilon_2+\varepsilon_1}\right) \tag{7.111}$$

and

$$\mathbf{r}'_{\text{im}} = \begin{cases}\mathbf{r}', & z>0,\\ \mathbf{r}'-2a\,\hat{\mathbf{z}}, & z<0.\end{cases} \tag{7.112}$$

Thus, prescribe an algorithm to determine the electric potential for planar dielectrics–a method of images for planar dielectrics.

(d) Determine the electric field to be

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_1}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{q_{\text{im}}}{4\pi\varepsilon_1}\frac{\mathbf{r}-\mathbf{r}'_{\text{im}}}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|^3}. \tag{7.113}$$

Draw the electric field lines for $\varepsilon_2<\varepsilon_1$, and compare it with the electric field lines for $\varepsilon_2>\varepsilon_1$.

(e) By evaluating the ratios

$$\frac{E_x(x,y,+\delta)}{E_x(x,y,-\delta)}, \quad \frac{E_y(x,y,+\delta)}{E_y(x,y,-\delta)}, \quad \frac{E_z(x,y,+\delta)}{E_z(x,y,-\delta)}, \tag{7.114}$$

determine the boundary conditions satisfied by the electric field lines. This is the Snell's law for the electric field lines. Note that the Snell's law for refraction is expressed in terms of the propagation vector of a plane wave, which is perpendicular to the electric field lines.

(f) A perfect conductor (in the static limit) is a dielectric material with a very high dielectric constant ($\varepsilon \to \infty$). Consider the extreme limit

$$\varepsilon_1 < \varepsilon_2 \to \infty \tag{7.115}$$

and

$$\varepsilon_2 < \varepsilon_1 \to \infty. \tag{7.116}$$

Examine these cases critically. Compare your results with the method of images for perfect conductors.

5. (**20 points.**)

(a) Find the solution to the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z} + \varepsilon(z)k_\perp^2\right] g(z,z';k_\perp) = \delta(z-z') \tag{7.117}$$

when

$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 & a < z. \end{cases} \tag{7.118}$$

for the case $a < z'$. Look for solution that is zero at $z = \pm\infty$.

(b) Consider a semi-infinite dielectric slab described by

$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 > \varepsilon_2 & a < z. \end{cases} \tag{7.119}$$

A point charge q described by

$$\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{7.120}$$

is embedded at position $\mathbf{r}'$ (with $a < z'$) on one side of the interface.

i. Show that the electric potential is given in terms of the Green's function by

$$\phi(\mathbf{r}) = qG(\mathbf{r},\mathbf{r}'), \tag{7.121}$$

where the Green's function satisfies

$$\boldsymbol{\nabla} \cdot \varepsilon(z)\boldsymbol{\nabla} G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{7.122}$$

Using the solution for the reduced Green's function $g(z,z';k_\perp)$ find the expression for the electric potential to be given by

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_1}\dfrac{1}{|\mathbf{r}-\mathbf{r}'|} + \dfrac{q}{4\pi\varepsilon_1}\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{1}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|}, & a < z, \\[2ex] \dfrac{2q}{4\pi(\varepsilon_1+\varepsilon_2)}\dfrac{1}{|\mathbf{r}-\mathbf{r}'|}, & z < a, \end{cases} \tag{7.123}$$

where $\mathbf{r}'_{\text{im}} = \mathbf{r}' - 2(z'-a)\hat{\mathbf{z}}$.

ii. Using $\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\phi(\mathbf{r})$ find the expression for the electric field as

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_1}\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} + \dfrac{q}{4\pi\varepsilon_1}\dfrac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\dfrac{\mathbf{r}-\mathbf{r}'_{\text{im}}}{|\mathbf{r}-\mathbf{r}'_{\text{im}}|^3}, & a < z, \\ \dfrac{2q}{4\pi(\varepsilon_1+\varepsilon_2)}\dfrac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}, & z < a. \end{cases} \tag{7.124}$$

iii. Draw the electric field lines for this configuration ($\varepsilon_2 < \varepsilon_1$).

iv. Investigate the continuity in the components of electric field at the interface by evaluating the following:

$$E_x(x,y,a+\delta) - E_x(x,y,a-\delta) = ?, \tag{7.125}$$

$$E_y(x,y,a+\delta) - E_y(x,y,a-\delta) = ?, \tag{7.126}$$

$$\varepsilon_1 E_z(x,y,a+\delta) - \varepsilon_2 E_z(x,y,a-\delta) = ?. \tag{7.127}$$

6. (**20 points.**) A monochromatic plane wave is characterized by the direction of propagation $\mathbf{k}$, the electric field $\mathbf{E}$, and the magnetic field $\mathbf{B}$, that satisfy

$$\mathbf{k}\cdot\mathbf{E} = 0, \quad \mathbf{k}\cdot\mathbf{B} = 0, \quad \mathbf{k}\times\mathbf{E} = \omega\mathbf{B}, \quad \mathbf{k}\times\mathbf{B} = -\frac{\omega}{c^2}\mathbf{E}. \tag{7.128}$$

These equations further imply $k = \omega/c$. At the interface of two materials, Snell's law of refraction states that the direction of propagation $\mathbf{k}$ bends towards the normal vector to the interface when the plane wave goes from a region of lower refractive index to a region of higher refractive index. Verify that the direction of the electric field bends away from the normal vector to the interface in the same scenario.

7. (**10 points.**) Consider a semi-infinite dielectric slab described by

$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 > \varepsilon_2 & a < z. \end{cases} \tag{7.129}$$

Find the expression for the electric potential due to a point dipole $\mathbf{d}$ placed at $\mathbf{r}'$ (with $a < z'$).
Hint: The charge density for a point dipole is

$$\rho(\mathbf{r}) = -\mathbf{d}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{7.130}$$

8. (**10 points.**) A perfectly conducting plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = d\,\hat{\mathbf{z}}$. Determine the direction and magnitude of electric field at $\mathbf{r} = d\,\hat{\mathbf{x}} + 2d\,\hat{\mathbf{z}}$.

9. (**10 points.**) A positive charge q and a negative charge q, a distance d apart from each other, are placed a distance $d/2$ away from a perfectly conducting plate. Determine the electrostatic force on the conductor?

Chapter 8

Cylindrical geometry

8.1 Bessel functions

1. (**10 points.**) Using the series representation for Bessel functions,

$$J_m(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(m+n)!} \left(\frac{t}{2}\right)^{m+2n}, \tag{8.1}$$

prove the relation

$$J_m(t) = (-1)^m J_{-m}(t). \tag{8.2}$$

Hint: Break the sum on n into two parts. Note that the gamma function $\Gamma(z)$, which generalizes the factorial,

$$n! = \Gamma(n+1), \qquad \Gamma(z+1) = z\Gamma(z), \tag{8.3}$$

beyond positive integers, satisfies

$$\frac{1}{\Gamma(-k)} = 0 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{8.4}$$

2. (**20 points.**) Use the integral representation of $J_m(t)$,

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}, \tag{8.5}$$

to prove the recurrence relations

$$2\frac{d}{dt} J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{8.6a}$$

$$2\frac{m}{t} J_m(t) = J_{m-1}(t) + J_{m+1}(t). \tag{8.6b}$$

3. (**10 points.**) Using the recurrence relations of Eq. (8.6), show that

$$\left(-\frac{d}{dt} + \frac{m-1}{t}\right)\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = \left(\frac{d}{dt} + \frac{m+1}{t}\right)\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_m(t) \tag{8.7}$$

and from this derive the differential equation satisfied by $J_m(t)$.

4. (**20 points.**) Using the recurrence relations,

$$2\frac{d}{dt}J_m(t) = J_{m-1}(t) - J_{m+1}(t), \tag{8.8a}$$

$$2\frac{m}{t}J_m(t) = J_{m-1}(t) + J_{m+1}(t), \tag{8.8b}$$

satisfied by the Bessel functions, derive the 'ladder' operations satisfied by the Bessel functions,

$$\left(\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m-1}(t), \tag{8.9}$$

$$\left(-\frac{d}{dt} + \frac{m}{t}\right) J_m(t) = J_{m+1}(t). \tag{8.10}$$

In quantum mechanics a ladder operator is a raising or lowering operator that transforms eigenfunctions by increasing or decreasing the eigenvalue.

5. (**20 points.**) Bessel function of zeroth order is defined by the integral representation

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}. \tag{8.11}$$

Verify that $J_0(t)$ is indeed a real function by showing that

$$\mathrm{Im}\,[J_0(t)] = 0. \tag{8.12}$$

6. (**20 points.**) Bessel function of zeroth order is defined by the integral representation

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}. \tag{8.13}$$

Verify, by substitution, that it satisifes the differential equation

$$\left[\frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt} + 1\right] J_0(t) = 0. \tag{8.14}$$

7. (**10 points.**) The Bessel functions $J_m(t)$ are defined by the expression

$$i^m J_m(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi} e^{it\cos\alpha - im\alpha}. \tag{8.15}$$

   (a) Evaluate $J_0(0)$.

   (b) Evaluate $J_m(0)$ for $m \neq 0$.

8. (**20 points.**) Starting from

$$\frac{\delta(\rho-\rho')\delta(\phi-\phi')}{\rho} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')}\int_0^\infty k_\perp dk_\perp J_m(k_\perp\rho)J_m(k_\perp\rho') \tag{8.16}$$

show that

$$\frac{\delta(\rho-\rho')}{\rho} = \int_0^\infty k_\perp dk_\perp J_m(k_\perp\rho)J_m(k_\perp\rho'). \tag{8.17}$$

Hint: Multiply the first equation by $e^{-im'(\phi-\phi')}$ on both sides, and integrate with respect to ϕ. Use the property of δ-function on the left hand side, and

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(m-m')(\phi-\phi')} = \delta_{mm'} \tag{8.18}$$

on the right hand side.

8.2 Modified Bessel functions

1. (**10 points.**) Starting from

$$\boldsymbol{\nabla} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{8.19}$$

show that

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}, \tag{8.20}$$

where (ρ, ϕ, z) are the cylindrical coordinates.

2. Show that

$$\boldsymbol{\nabla}\cdot\varepsilon(\rho)\boldsymbol{\nabla} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\varepsilon(\rho)\rho\frac{\partial}{\partial\rho} + \frac{\varepsilon(\rho)}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \varepsilon(\rho)\frac{\partial^2}{\partial z^2}, \tag{8.21}$$

where (ρ, ϕ, z) are the cylindrical coordinates.

3. (**10 points.**) Integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$ are

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{8.22a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\, \cos m\phi\, e^{t\cos\phi}. \tag{8.22b}$$

   (a) Using Mathematica (or your favourite graphing tool) plot $K_0(t), K_1(t), K_2(t)$ and $I_0(t), I_1(t), I_2(t)$ on the same plot. (Please do not submit hand sketched plots.)

   (b) Refer Chapter 10 of Digital Library of Mathematical Functions,

   https://dlmf.nist.gov/10

   for a comprehensive resource.

4. (**10 points.**) Show that the integral representations for the modified Bessel functions, $I_m(t)$ and $K_m(t)$, for integer m and $0 \le t < \infty$,

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{8.23a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\, \cos m\phi\, e^{t\cos\phi}. \tag{8.23b}$$

satisfies the differential equation for modified Bessel functions,

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{8.24}$$

Hint: Integrate by parts, after identifying

$$\left(t\cosh\theta - t^2\sinh^2\theta\right) e^{-t\cosh\theta} = -\frac{d^2}{d\theta^2}e^{-t\cosh\theta}, \tag{8.25a}$$

$$\left(t\cos\phi - t^2\sin^2\phi\right) e^{t\cos\phi} = -\frac{d^2}{d\phi^2}e^{t\cos\phi}. \tag{8.25b}$$

5. (**20 points.**) The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{8.26}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(t)K_m'(t) - K_m(t)I_m'(t) = -\frac{C}{t}, \tag{8.27}$$

where

$$I_m'(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K_m'(t) \equiv \frac{d}{dt}K_m(t). \tag{8.28}$$

Further, determine the value of the constant C on the right hand side of Eq. (8.27) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{8.29}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{8.30}$$

6. (**20 points.**) Using the integral representations for the modified Bessel functions,

$$K_m(t) = \int_0^\infty d\theta\, \cosh m\theta\, e^{-t\cosh\theta}, \tag{8.31a}$$

$$I_m(t) = \int_0^\pi \frac{d\phi}{\pi}\, \cos m\phi\, e^{t\cos\phi}, \tag{8.31b}$$

derive the asymptotic forms for large t,

$$K_m(t) \xrightarrow{1\ll t} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}, \tag{8.32a}$$

$$I_m(t) \xrightarrow{1\ll t} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}. \tag{8.32b}$$

Hint: The contributions to the integral are dominated near the lower limit, so use $\cosh m\theta \sim 1 + \frac{1}{2}m^2\theta^2$ and $\cos m\phi \sim 1 - \frac{1}{2}m^2\phi^2$.

8.3 Cylindrical Green's functions

8.3.1 Free Green's function

1. (**20 points.**) The free Green's function satisfies

$$-\nabla^2 G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{8.33}$$

The free Green's function is the electric potential at $\mathbf{r}$ due to a point charge of unit magnitude at $\mathbf{r}'$,

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}')|}. \tag{8.34}$$

To derive a representation for the free Green's function suitable for cylindrically symmetric configurations, we require translational symmetry in the z and rotational symmetry in ϕ, in terms cylindrical coordinates (ρ,ϕ,z),

$$G(\mathbf{r},\mathbf{r}') = G(\rho,\rho',\phi-\phi',z-z'). \tag{8.35}$$

In cylindrical coordinates we have

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \delta(\rho-\rho')\frac{\delta(\phi-\phi')}{\rho}\delta(z-z'). \tag{8.36}$$

Use Fourier transformations to write

$$G(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')} g_m(\rho,\rho';k_z) \tag{8.37}$$

and

$$\delta^{(3)}(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')}{\rho} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \sum_{m=-\infty}^{\infty} \frac{1}{2\pi} e^{im(\phi-\phi')}. \tag{8.38}$$

Then, show that the reduced free Green's function satisfies the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{8.39}$$

2. (**40 points.**) The cylindrical free Green's function satisfies

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k_z^2\right] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{8.40}$$

(a) Integrate Eq. (8.40) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{8.41a}$$

$$\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{8.41b}$$

(b) Let us further require that

$$g_m(0,\rho';k_z) \text{ is finite}, \tag{8.42a}$$

$$g_m(\infty,\rho';k_z) = 0. \tag{8.42b}$$

(c) Recall the Wronskian

$$I_m(t)K_m'(t) - I_m'(t)K_m(t) = -\frac{1}{t}. \tag{8.43}$$

(d) Construct the solution to have the form

$$g_m(\rho,\rho') = \begin{cases} A\, I_m(k_z\rho) + B\, K_m(k_z\rho), & 0 \le \rho < \rho', \\ C\, I_m(k_z\rho) + D\, K_m(k_z\rho), & \rho' < \rho < \infty. \end{cases} \tag{8.44}$$

Derive the solution

$$g_m(\rho,\rho') = I_m(k_z\rho_<)K_m(k_z\rho_>), \tag{8.45}$$

where $\rho_< = \text{Minimum}(\rho,\rho')$ and $\rho_> = \text{Maximum}(\rho,\rho')$.

3. (**20 points.**) Verify by substitution that

$$\begin{aligned} g_m(\rho,\rho';k) &= I_m(k\rho_<)K_m(k\rho_>) \\ &= \theta(\rho'-\rho)I_m(k\rho)K_m(k\rho') + \theta(\rho-\rho')I_m(k\rho')K_m(k\rho) \end{aligned} \tag{8.46}$$

satisfies the differential equation

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{m^2}{\rho^2} + k^2\right] g_m(\rho,\rho';k) = \frac{\delta(\rho-\rho')}{\rho}. \tag{8.47}$$

Hint: Use the identity $d\theta(x)/dx = \delta(x)$.

Qualitatively sketch the electric field lines of a point charge placed (off centered) inside a conducting cylinder. Next, sketch the electric potential of a point charge inside a conducting cylinder. Show both the constant z cross section and constant x cross section.

4. (**20 points.**) Verify by substitution that

$$g_m(t,t') = I_m(t_<)K_m(t_>) \tag{8.48}$$

satisfies the differential equation

$$\Big[-\frac{1}{t}\frac{\partial}{\partial t}t\frac{\partial}{\partial t} + \frac{m^2}{t^2} + 1\Big] g_m(t,t') = \frac{\delta(t-t')}{t}. \tag{8.49}$$

8.3.2 Green's function for (inside) a perfectly conducting cylinder

1. (**20 points.**) The cylindrical Green's function satisfies

$$\Big[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2} + k_z^2\Big] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{8.50}$$

Inside a perfectly conducting cylinder we require

$$g_m(0,\rho';k_z) \ \text{is finite}, \tag{8.51a}$$
$$g_m(a,\rho';k_z) = 0. \tag{8.51b}$$

This shields all of (electrostatics related) physics between the plates from outside. Thus, we have

$$0 \le \rho \le a, \tag{8.52a}$$
$$0 \le \rho' \le a, \tag{8.52b}$$

Integrate Eq. (8.57) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{8.53a}$$

$$\rho\frac{\partial}{\partial \rho} g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{8.53b}$$

Recall the Wronskian

$$I_m(t)K_m'(t) - I_m'(t)K_m(t) = -\frac{1}{t}. \tag{8.54}$$

Construct the solution to have the form

$$g_m(\rho,\rho') = \begin{cases} A\, I_m(k_z\rho) + B\, K_m(k_z\rho), & 0 \le \rho < \rho', \\ C\, I_m(k_z\rho) + D\, K_m(k_z\rho), & \rho' < \rho < a. \end{cases} \tag{8.55}$$

Derive the solution

$$g_m(\rho,\rho') = I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{K_m(k_z a)}{I_m(k_z a)} I_m(k_z\rho) I_m(k_z\rho'). \tag{8.56}$$

2. (**80 points.**) The cylindrical Green's function satisfies

$$\Big[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\varepsilon(\rho)\rho\frac{\partial}{\partial \rho} + \varepsilon(\rho)\frac{m^2}{\rho^2} + \varepsilon(\rho)k_z^2\Big] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}. \tag{8.57}$$

Consider a dielectric cylinder sorrounded by a perfectly conducting cylinder described by

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2 & \text{for} \quad \rho < a, \\ \infty & \text{for} \quad a < \rho. \end{cases} \tag{8.58}$$

(a) Integrate Eq. (8.57) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{8.59a}$$

$$\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -\frac{1}{\varepsilon_2}. \tag{8.59b}$$

(b) Using the property of a perfectly conducting cylinder we require

$$g_m(0, \rho'; k_z) \text{ is finite}, \tag{8.60a}$$

$$g_m(a, \rho'; k_z) = 0. \tag{8.60b}$$

(c) Recall the Wronskian

$$I_m(t)K'_m(t) - I'_m(t)K_m(t) = -\frac{1}{t}. \tag{8.61}$$

(d) Derive the solution

$$g_m(\rho, \rho') = \frac{1}{\varepsilon_2} I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{1}{\varepsilon_2}\frac{K_m(k_z a)}{I_m(k_z a)} I_m(k_z\rho)I_m(k_z\rho'). \tag{8.62}$$

8.3.3 Green's function for a dielectric cylinder

1. (**80 points.**) The cylindrical Green's function satisfies

$$\Big[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\varepsilon(\rho)\rho\frac{\partial}{\partial \rho} + \varepsilon(\rho)\frac{m^2}{\rho^2} + \varepsilon(\rho)k_z^2\Big] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \tag{8.63}$$

Consider a dielectric cylinder described by

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2 & \text{for} \quad \rho < a, \\ \varepsilon_1 & \text{for} \quad a < \rho. \end{cases} \tag{8.64}$$

(a) Integrate Eq. (8.63) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \tag{8.65a}$$

$$\varepsilon(\rho)\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \tag{8.65b}$$

(b) Integrate Eq. (8.63) around $\rho = a$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z)\Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0, \tag{8.66a}$$

$$\varepsilon(\rho)\rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z)\Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0. \tag{8.66b}$$

(c) For

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2, & \rho < a, \\ \varepsilon_1, & a < \rho, \end{cases} \tag{8.67}$$

derive the solution

$\underline{\rho' < a}$:

$$g_m(\rho,\rho';k_z) = \begin{cases} \dfrac{1}{\varepsilon_2} I_m(k_z\rho_<)K_m(k_z\rho_>) - \dfrac{1}{\varepsilon_2} I_m(k_z\rho)I_m(k_z\rho')\dfrac{K_a K_a'}{\Delta}, & \rho,\rho' < a, \\ \dfrac{1}{\varepsilon_1} I_m(k_z\rho_<)K_m(k_z\rho_>) - \dfrac{1}{\varepsilon_1} K_m(k_z\rho)I_m(k_z\rho')\dfrac{I_a' K_a}{\Delta}, & \rho' < a < \rho. \end{cases} \tag{8.68}$$

$\underline{a < \rho'}$:

$$g_m(\rho,\rho';k_z) = \begin{cases} \dfrac{1}{\varepsilon_2} I_m(k_z\rho_<)K_m(k_z\rho_>) - \dfrac{1}{\varepsilon_2} I_m(k_z\rho)K_m(k_z\rho')\dfrac{I_a K_a'}{\Delta}, & \rho < a < \rho', \\ \dfrac{1}{\varepsilon_1} I_m(k_z\rho_<)K_m(k_z\rho_>) - \dfrac{1}{\varepsilon_1} K_m(k_z\rho)K_m(k_z\rho')\dfrac{I_a I_a'}{\Delta}, & a < \rho,\rho'. \end{cases} \tag{8.69}$$

We used the definitions

$$\frac{1}{\Delta} = \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 I_a K_a' - \varepsilon_2 K_a I_a')}, \qquad I_a \equiv I_m(k_z a), \quad K_a \equiv K_m(k_z a). \tag{8.70}$$

(d) Show that in the perfect conductor limit

Inside the cylinder

$$g_m(\rho,\rho';k_z) = \frac{1}{\varepsilon_2} I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{1}{\varepsilon_2} I_m(k_z\rho)I_m(k_z\rho')\frac{K_a}{I_a}. \tag{8.71}$$

Outside the cylinder

$$g_m(\rho,\rho';k_z) = \frac{1}{\varepsilon_1} I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{1}{\varepsilon_1} K_m(k_z\rho)K_m(k_z\rho')\frac{I_a}{K_a}. \tag{8.72}$$

8.3.4 A point charge inside a cavity with perfectly conducting circular cylindrical boundary

1. (**50 points.**) Consider a point charge q placed on the axis of a perfectly conducting circular cylinder of radius a.

   (a) The relevant Maxwell equation is

   $$-\boldsymbol{\nabla}\cdot\varepsilon(\rho)\boldsymbol{\nabla}\phi(\mathbf{r}) = \rho(\mathbf{r}) \tag{8.73}$$

   with dielectric function

   $$\varepsilon(\rho) = \begin{cases} \varepsilon_0, & \rho < a, \\ \varepsilon_1 \to \infty, & a < \rho, \end{cases} \tag{8.74}$$

   and charge at origin

   $$\rho(\mathbf{r}) = q\delta^{(3)}(\mathbf{r}). \tag{8.75}$$

   The associated Green's function equation is

   $$-\boldsymbol{\nabla}\cdot\varepsilon_0\boldsymbol{\nabla}G(\mathbf{r},\mathbf{r}') = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{8.76}$$

   with solution

   $$G(\mathbf{r},\mathbf{r}') = \int_{-\infty}^{\infty}\frac{dk_z}{2\pi}e^{ik_z(z-z')}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\phi-\phi')}g_m(\rho,\rho';k_z), \tag{8.77}$$

   where

   $$g_m(\rho,\rho';k_z) = \frac{1}{\varepsilon_0}\left[I_m(k_z\rho_<)K_m(k_z\rho_>) - \frac{K_m(ka)}{I_m(ka)}I_m(k_z\rho_<)I_m(k_z\rho_>)\right]. \tag{8.78}$$

(b) Using the connection between the electric potential and the Green function,

$$\phi(\mathbf{r}) = q\, G(\mathbf{r}, \mathbf{r}_0), \tag{8.79}$$

where $\mathbf{r}_0 = \mathbf{0}$ is the position of the position of the charge q, that is chosen to be at the origin without any loss in generality, determine the electric potential to be

$$\phi(\mathbf{r}) = \frac{q}{2\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \left[K_0(k\rho) - \frac{K_0(ka)}{I_0(ka)} I_0(k\rho) \right], \tag{8.80}$$

where $\mathbf{r} = (\rho, \phi, z)$ is the observation point. Here a is the radius of the cylinder.

(c) Verify that the potential satisfies the boundary condition

$$\phi(\mathbf{a}) = 0 \tag{8.81}$$

on the inner surface of the conducting cylinder.

(d) Using the relation $\mathbf{E} = -\boldsymbol{\nabla}\phi$ evaluate the electric field on the inner surface of the conductor to be

$$\mathbf{E}(\mathbf{a}) = \hat{\boldsymbol{\rho}} \frac{q}{2\pi\varepsilon_0} \frac{1}{a} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ikz}}{I_0(ka)}. \tag{8.82}$$

Observe that the electric field is normal to the inner surface of the cylinder. Use the Wronskian.

(e) Using Gauss's law argue that the induced charge on the surface of a conductor is given using

$$\sigma(\phi, z) = \varepsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}\Big|_{\text{surface}}, \tag{8.83}$$

where $\hat{\mathbf{n}}$ is normal to the surface of conductor. Thus, determine the induced charge density on the inner surface of the cylinder to be

$$\sigma(\phi, z) = -\frac{q}{4\pi^2 a^2} \int_{-\infty}^{\infty} dt \frac{e^{it\frac{z}{a}}}{I_0(t)}. \tag{8.84}$$

(f) By integrating over the surface of the cylinder determine the total induced charge on the cylinder. Thus, find out if its magnitude is less than, equal to, or greater than, the charge q.
Solution: $-q$.

2. (**10 points.**) The radial part of the Green function inside a cavity with perfectly conducting circular cylindrical boundary of radius a is

$$g_m(\rho, \rho'; k_z) = I_m(k_z\rho_<) K_m(k_z\rho_>) - \frac{K_m(k_z a)}{I_m(k_z a)} I_m(k_z\rho) I_m(k_z\rho'), \tag{8.85}$$

where $0 \leq \rho, \rho' < a$. Here $\rho_< = \text{Min}(\rho, \rho')$, $\rho_> = \text{Max}(\rho, \rho')$, k_z is the Fourier variable for the z-coordinate and m is the Fourier variable for the angular coordinate ϕ. Evaluate $g_m(a, \rho'; k_z)$. Give a physical reasoning for your answer.

8.3.5 A point charge outside a perfectly conducting circular cylinder

1. (**50 points.**) Consider a point charge q placed a radial distance $\rho_0 > a$ away from the axis of a perfectly conducting cylinder. Here a is the radius of the cylinder.

(a) Using the connection between the electric potential and Green's function,

$$\phi(\mathbf{r}) = q\, G(\mathbf{r}, \mathbf{r}_0), \tag{8.86}$$

and the Green function for a perfectly conducting cylinder, derived in class, determine the electric potential to be

$$\phi(\mathbf{r}) = \frac{q}{\varepsilon_0} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \sum_{-\infty}^{\infty} \frac{1}{2\pi} e^{im\phi} \left[I_m(k\rho_<) K_m(k\rho_>) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho) K_m(k\rho_0) \right], \tag{8.87}$$

where $\mathbf{r} = (\rho, \phi, z)$ and the position of the point charge $\mathbf{r}_0$ is chosen to be $(\rho_0, 0, 0)$.

(b) Verify that the potential satisfies the boundary condition

$$\phi(\mathbf{a}) = 0 \tag{8.88}$$

on the surface of the conducting cylinder.

(c) Using the relation $\mathbf{E} = -\boldsymbol{\nabla}\phi$ evaluate the electric field on the surface of the conductor to be

$$\mathbf{E}(\mathbf{a}) = -\hat{\boldsymbol{\rho}} \frac{q}{\varepsilon_0} \frac{1}{a} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikz} \sum_{-\infty}^{\infty} \frac{1}{2\pi} e^{im\phi} \frac{K_m(k\rho_0)}{K_m(ka)}. \tag{8.89}$$

Note that the electric field is normal to the surface of the cylinder.

(d) Using Gauss's theorem we can argue that the induced charge on the surface of a conductor is given using

$$\sigma(\phi, z) = \varepsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}\Big|_{\text{surface}}, \tag{8.90}$$

where $\hat{\mathbf{n}}$ is normal to the surface of conductor. Thus, determine the induced charge density $\sigma(\phi, z)$ on the surface of the cylinder.

(e) By integrating over the surface of the cylinder determine the total induced charge on the cylinder. Thus, find out if its magnitude is less than, equal to, or greater than, the charge q.

2. (**20 points.**) The modified Bessel function of zeroth order has the following asymptotic form near $t = 0$,

$$K_0(t) \sim \ln \frac{2}{t} - \gamma, \quad t \ll 1, \tag{8.91}$$

where $\gamma = 0.1159\ldots$ is the Euler's constant. Evaluate the limit

$$\lim_{t \to 0} \frac{K_0(at)}{K_0(t)} \tag{8.92}$$

for positive real a.

3. (**20 points.**) The radial part of the Green function outside a perfectly conducting right circular cylinder of radius a is

$$g_m(\rho, \rho'; k) = I_m(k\rho_<) K_m(k\rho_>) - \frac{I_m(ka)}{K_m(ka)} K_m(k\rho) K_m(k\rho'), \tag{8.93}$$

where $a \le \rho, \rho' < \infty$. Here $\rho_< = \text{Min}(\rho, \rho')$, $\rho_> = \text{Max}(\rho, \rho')$, k is the Fourier variable for the z-coordinate and m is the Fourier variable for the angular coordinate ϕ. Evaluate $g_m(a, \rho'; k)$. Give a physical reasoning for your answer.

4. (**20 points.**) The free Green's function represents the electric potential of a unit point charge,

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\varepsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|}. \tag{8.94}$$

For a point charge placed at the origin we have, choosing $\mathbf{r}'$ at the origin, in cylindrical coordinates

$$G_0(\mathbf{r}, 0) = \frac{1}{4\pi\varepsilon_0} \frac{1}{\sqrt{\rho^2 + z^2}}. \tag{8.95}$$

The free Green's function also has the representation

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(z-z')}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\phi-\phi')}I_m(k\rho_<)K_m(k\rho_>). \tag{8.96}$$

Using Eq. (8.96), determine $G_0(\mathbf{r},0)$ in terms of a single integral. That is, evaluate the sum for this case.

8.3.6 A line charge outside a perfectly conducting cylinder

1. (**60 points.**) Consider a wire of infinite length and negligible thickness outside a cylindrical cavity with perfectly conducting walls. The cylindrical cavity in cross section forms a circle of radius a.

   (a) The Green's function outside such a perfectly conducting cylinder is

$$\begin{aligned} G(\mathbf{r},\mathbf{r}') &= \frac{1}{\varepsilon_0}\int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(z-z')}\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}e^{im(\phi-\phi')} \\ &\quad\times\left[I_m(k\rho_<)K_m(k\rho_>) - \frac{I_m(ka)}{K_m(ka)}K_m(k\rho)K_m(k\rho')\right]. \end{aligned} \tag{8.97}$$

   For a wire with uniform charge per unit length we have

$$\lambda = \frac{dq}{dz'}. \tag{8.98}$$

   The electric potential outside such a cylinder is given by

$$\phi(\boldsymbol{\rho}) = \lambda\,\mathcal{G}(\boldsymbol{\rho},\boldsymbol{\rho}') = \lambda\int_{-\infty}^{\infty}dz'\,G(\mathbf{r},\mathbf{r}'). \tag{8.99}$$

   Show that

$$\lambda\mathcal{G}(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{\lambda}{2\pi\varepsilon_0}\lim_{k\to0}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\left[I_m(k\rho_<)K_m(k\rho_>) - \frac{I_m(ka)}{K_m(ka)}K_m(k\rho)K_m(k\rho')\right]. \tag{8.100}$$

   (b) Using $I_{-m}(t) = I_m(t)$ for integer m, $K_{-m}(t) = K_m(t)$ for any m, and the limiting forms

$$I_0(t) \sim 1, \tag{8.101a}$$

$$K_0(t) \sim -\ln t, \tag{8.101b}$$

$$I_m(t) \sim \frac{1}{m!}\left(\frac{t}{2}\right)^m, \tag{8.101c}$$

$$K_m(t) \sim \frac{(m-1)!}{2}\left(\frac{2}{t}\right)^m, \tag{8.101d}$$

   show that

$$\lim_{k\to0}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}I_m(k\rho_<)K_m(k\rho_>) \sim -\ln k|\boldsymbol{\rho}-\boldsymbol{\rho}'|, \tag{8.102a}$$

$$\lim_{k\to0}\sum_{m=-\infty}^{\infty}e^{im(\phi-\phi')}\frac{I_m(ka)}{K_m(ka)}K_m(k\rho)K_m(k\rho') \sim -\ln\frac{\rho'}{a} - \ln k\left|\boldsymbol{\rho} - \frac{a^2}{\rho'^2}\boldsymbol{\rho}'\right|. \tag{8.102b}$$

   Thus, show that

$$\lambda\mathcal{G}(\boldsymbol{\rho},\boldsymbol{\rho}') = -\frac{\lambda}{2\pi\varepsilon_0}\ln k|\boldsymbol{\rho}-\boldsymbol{\rho}'| + \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\rho'}{a} + \frac{\lambda}{2\pi\varepsilon_0}\ln k\left|\boldsymbol{\rho} - \frac{a^2}{\rho'^2}\boldsymbol{\rho}'\right|, \tag{8.103}$$

   where k could be interpreted as the inverse of length of the cylinder. Interpret this expression in terms of images.

(c) Verify that

$$\phi(\mathbf{a}) = 0. \tag{8.104}$$

Here $\mathbf{a} = a\hat{\boldsymbol{\rho}}$.

(d) Show that the electric field $\mathbf{E} = -\boldsymbol{\nabla}\phi$ is

$$\mathbf{E}(\boldsymbol{\rho}) = \frac{\lambda}{2\pi\varepsilon_0}\frac{(\boldsymbol{\rho}-\boldsymbol{\rho}')}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2} - \frac{\lambda}{2\pi\varepsilon_0}\frac{\left(\boldsymbol{\rho}-\frac{a^2}{\rho'^2}\boldsymbol{\rho}'\right)}{\left|\boldsymbol{\rho}-\frac{a^2}{\rho'^2}\boldsymbol{\rho}'\right|^2}. \tag{8.105}$$

Determine $\mathbf{E}(\mathbf{a})$.

(e) Induced charge density on the surface of the conductor is given by

$$\sigma(\phi) = \varepsilon_0\hat{\mathbf{a}}\cdot\mathbf{E}(\mathbf{a}). \tag{8.106}$$

Show that

$$\sigma(\phi) = -\frac{\lambda}{2\pi a}\frac{({\rho'}^2 - a^2)}{\left[{\rho'}^2 - 2a\rho'\cos(\phi-\phi') + a^2\right]}. \tag{8.107}$$

(f) Show that the total induced charge per unit length is equal to

$$\int_0^{2\pi} a d\phi\, \sigma(\phi) = -\lambda. \tag{8.108}$$

Use the integral

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{1}{(1-t\cos\phi)} = \frac{1}{\sqrt{1-t^2}}, \qquad |t|<1. \tag{8.109}$$

(g) Refer to related dicussions in problems 2.8, 2.11, 2.17, and 2.18 in Jackson.

8.3.7 A line charge inside a cylindrical cavity with perfectly conducting walls

Chapter 9

Spherical geometry

9.1 Spherical geometry: Method of images

1. (**20 points.** Take home exercise, to be submitted during exam.)
   Consider a spherical cavity of radius a with perfectly conducting walls that is grounded. The inside of the cavity is described by vacuum properties ε_0. A point charge q is placed inside the cavity.

   (a) Using method of images determine the magnitude and position of the fictitious image charge that will simulate the boundary conditions of a perfect conductor on the inner surface of the conductor.

   (b) Write down the total electric potential due to the original charge (inside the sphere) and the image charge. Thus determine the electric potential everywhere inside the spherical conductor.

   (c) Determine the induced charge density on the inner surface of the spherical conductor.

   (d) Integrating the induced charge density over the inner surface of the conductor determine the total induced charge. Thus, find out if the total induced charge equals the image charge?

Chapter 10

Magnetostatics

10.1 Magnetic force

1. (**20 points.**) A charged particle initially moving with constant speed v enters a region of magnetic field $\mathbf{B}$ pointing into the page. It is deflected as shown in Fig. 10.1.

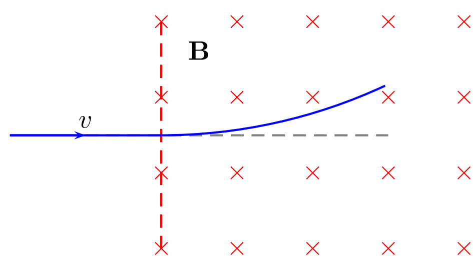

Figure 10.1: Problem 1

   (a) Is the charge on the particle positive or negative?

   (b) What curve characterizes the path of the deflected particle?

2. (**30 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{v}\times\mathbf{B}. \tag{10.1}$$

   Choose $\mathbf{B}$ along the positive z-axis and solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{10.2a}$$

$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + v_0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{10.2b}$$

   Verify that the solution describes a circle of radius R with center at position $R\,\hat{\mathbf{i}}$. Find R. For the same initial velocity does an electron or a proton have a larger radii.

3. (**15 points.**) The magnetic field of an infinitely long straight wire carrying a steady current I is given by, (assume wire on z-axis,)

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{10.3}$$

where $\rho = \sqrt{x^2+y^2}$ is the closest distance of point $\mathbf{r}$ from the wire. The Lorentz force on a particle of charge q and mass m is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}. \tag{10.4}$$

In the absence of an electric field, qualitatively, describe the motion of a positive charge with an initial velocity in the z-direction. In particular, investigate if the particle will attain a speed in the ϕ-direction. Thus answer whether the charge will go around the wire?

4. (**30 points.**) Motion of a charged particle of mass m and charge q in a uniform magnetic field $\mathbf{B}$ and a uniform electric field $\mathbf{E}$ is governed by

$$m\frac{d\mathbf{v}}{dt} = q\,\mathbf{E} + q\,\mathbf{v}\times\mathbf{B}. \tag{10.5}$$

Choose $\mathbf{B}$ along the z-axis and $\mathbf{E}$ along the y-axis,

$$\mathbf{B} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + B\,\hat{\mathbf{k}}, \tag{10.6a}$$
$$\mathbf{E} = 0\,\hat{\mathbf{i}} + E\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{10.6b}$$

Solve this vector differential equation to determine the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ of the particle as a function of time, for initial conditions

$$\mathbf{x}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \tag{10.7a}$$
$$\mathbf{v}(0) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}. \tag{10.7b}$$

Verify that the solution is a cycloid characterized by the equations

$$x(t) = R(\omega_c t - \sin\omega_c t), \tag{10.8a}$$
$$y(t) = R(1 - \cos\omega_c t). \tag{10.8b}$$

where

$$R = \frac{E}{B\omega_c}, \qquad \omega_c = \frac{qB}{m}. \tag{10.9}$$

The particle moves as though it were a point on the rim of a wheel of radius R perfectly rolling (without sliding or slipping) with angular speed ω_c along the x-axis. It satisfies the equation of a circle of radius R whose center $(vt, R, 0)$ travels along the x-direction at constant speed v,

$$(x - vt)^2 + (y - R)^2 = R^2, \tag{10.10}$$

where $v = \omega_c R$.

Think: Initially the charge was at rest, implying zero initial momentum. The final velocity of the particle (on an average) is governed by the speed $v = \omega_c R = E/B$ along the x-axis. Observe that the field configuration has a momentum density of $\mathbf{G} = \varepsilon_0 \mathbf{E}\times\mathbf{B}$. and energy density $U = \varepsilon_0 E^2/2 + B^2/2\mu_0$. The ratio, for $\mathbf{E}\cdot\mathbf{B} = 0$,

$$\frac{U}{G} = \frac{1}{2}\frac{E}{B}\left[1 + \frac{c^2}{(E/B)^2}\right] \tag{10.11}$$

is a measure of velocity.

5. (**20 points.**) (Based on Griffiths 4th ed. problem 5.45.)
In 1897, J. J. Thompson 'discovered' the electron.

   (a) Describe briefly how this discovery influenced the model of an atom in those days.

   (b) Apparently, the experiment involved the measures of the radius of curvature R of the beam. Suggest a convenient and a reasonably precise method to measure R of a beam. Assume you have the technology available in the times of year 1900. Next, assume you have the technology available in the times of year 2019.

6. (**20 points.**) A charged particle in a magnetic field goes in circles (or in helices). Recall that positron is the antiparticle of electron. Describe the motion of a positron in a magnetic field, and contrast it to that of an electron in a magnetic field. How will the ionization track of electron and positron differ in a bubble chamber? For example, refer to the picture at 34:21 minute in the lecture by Frank Close, part of

   Christmas Lectures, 1993.

7. (**20 points.**) (Based on Griffiths 4th ed. problem 5.45.)
   A (hypothetical) stationary magnetic monopole q_m held fixed at the origin will have a magnetic field

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q_m}{r^2} \hat{\mathbf{r}}, \tag{10.12}$$

   because $\boldsymbol{\nabla} \cdot \mathbf{B} \neq 0$ anymore. Consider the motion of a particle with mass m and electric charge q_e in the field of this magnetic monopole.

   (a) Draw the magnetic field lines of the stationary magnetic monopole.

   (b) Using

$$\mathbf{F} = q_e \mathbf{v} \times \mathbf{B} \tag{10.13}$$

   derive the equation of motion for the electric charge to be

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{r} \frac{\mu_0}{4\pi} \frac{q_e q_m}{r^3} \frac{1}{m}, \tag{10.14}$$

   where $\mathbf{v}$ is the velocity of the electric charge q_e.

   (c) Recall that the motion of an electric charge in a uniform magnetic field implies circular (or helical) motion, which in turn implies that the speed $v = |\mathbf{v}|$ is a constant of motion. Show that the speed $v = |\mathbf{v}|$ is a constant of motion even for the motion of an electric charge in the field of a magnetic monopole. That is, show that

$$\frac{dv}{dt} = 0. \tag{10.15}$$

   (Hint: Show that $v^2 = \mathbf{v} \cdot \mathbf{v}$ is a constant of motion. Use $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.) However, the motion is not circular. Nevertheless, it is exactly solvable and the orbit is unbounded and lies on a right circular half-cone with vertex at the monopole. The comments following Eq. (10.15) are for your information and need not be proved here.

8. (**20 points.**) The force $d\mathbf{F}$ on an infinitely small line element $d\mathbf{l}$ of wire, carrying steady current I, placed in a magnetic field $\mathbf{B}$, is

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}. \tag{10.16}$$

   This involves the correspondence

$$q\mathbf{v} \to I d\mathbf{l} \tag{10.17}$$

   for the flow of charge, representing current, in the wire. Consider a wire segment of arbitrary shape (in the shape of a curve C) with one end at the origin and the other end at the tip of vector $\mathbf{L}$. The total force on the segment of wire is given by the line integral

$$\mathbf{F} = \int_{\mathbf{0}\,(\text{path}\, C)}^{\mathbf{L}} I d\mathbf{l} \times \mathbf{B}. \tag{10.18}$$

   Evaluate the total force on a closed loop of wire (of arbitrary shape and carrying steady current I) when it is placed in a uniform magnetic field? Check your result for a loop of wire in the shape of a square in a uniform magnetic field.

10.2 Basic magnetic field configurations

1. (**20 points.**) The magnetic field at a distance R from a wire of infinite extent carrying a steady current I is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{2I}{R}\hat{\boldsymbol{\phi}}, \tag{10.19}$$

where the direction of $\hat{\boldsymbol{\phi}}$ is given by the right-hand rule. Find the magnetic field at point o in Fig. 1 in terms of distances a and b and current I.

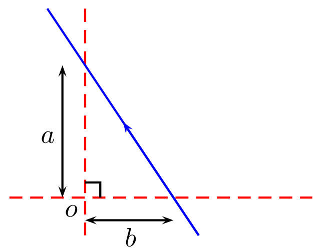

Figure 10.2: Problem 1

2. (**20 points.**) A steady current I flows through a wire shown in Fig. 2. Find the magnitude and direction of magnetic field at point P. You are given the magnitude of the magnetic field due to an infinite length

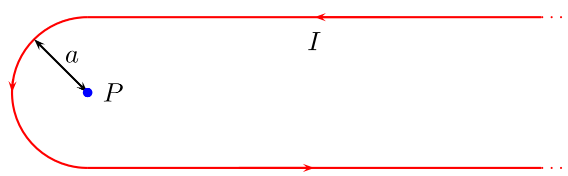

Figure 10.3: Problem 2.

of wire at distance ρ, and a circular loop of wire of radius R at the center of loop, to be

$$B_{\infty\text{-wire}} = \frac{\mu_0 I}{2\pi\rho} \qquad B_{\text{loop}} = \frac{\mu_0 I}{2R}. \tag{10.20}$$

3. (**20 points.**) A way of determining the sign of charge carriers in a conductor is by means of the Hall effect. A magnetic field $\mathbf{B}$ is applied perpendicular to the direction of current flow in a conductor, and as a consequence a transverse voltage drop appears across the conductor. If d is the transverse length of the conductor, and v is the average drift speed of the charge carriers, show that the voltage, in magnitude, is

$$V = vBd. \tag{10.21}$$

Estimate this potential drop (magnitude and direction) for a car driving towards North in the Northern hemisphere. How will the answer differ in the Southern hemisphere?

10.3 Ampere's law

1. (**0 points.**) Keywords: Magnetostatics (Chap. 5, Griffiths 4th edition), Ampere's law (Sec. 5.2, Griffiths 4th edition), Bio-Savart (Sec. 5.2, Griffiths 4th edition).

2. (**20 points.**) A steady current I flows down an infinitely long cylindrical wire of radius a. Using Ampère's law find the magnetic field, both inside and outside the wire, if the current is uniformly distributed over the outside surface of the wire.

3. (**30 points.**) Using Ampere's law determine the magnetic field inside and outside a solenoid of radius R and of infinite extent in the directions of its symmetry axis.

4. (**20 points.**) A solenoid consists of a current carrying wire in the shape of a coil wound into a tightly packed helix on the surface of a circular cylinder of radius R. The magnetic field of a solenoid is characterized by the current I in the wire and the number of turns per unit length n of the coil. Remarkably, the magnetic field inside a solenoid is independent of the radius R. Using Ampere's law deduce the expression for the magnetic field inside and outside the solenoid. Use the following based on symmetry.

   (a) The direction of th magnetic field is along the symmetry axis of the solenoid.

   (b) The magnitude of the magnetic field is independent of the associated cylindrical coordinates z and ϕ. Thus,
   $$\mathbf{B} = \hat{\mathbf{z}} B(\rho). \tag{10.22}$$

   (c) The magnetic field goes to zero as $\rho \to \infty$.

5. (**20 points.**) A solenoid has the geometry of a right circular cylinder of radius a and height extending to infinity on both ends. Using Ampere's law show that the magnetic field is uniform inside the solenoid and zero outside the solenoid. How does this result change for a solenoid of arbitrary cross section. Refer literature, and critically assess their ideas. Give a very brief report of your assessment.

6. (**20 points.**) An infinitely long wire of circular cross section radius a carries a steady current I. Another wire, in the form of a cylindrical shell and concentric to the first wire, has inner radius b and outer radius c, such that $a < b < c$. The region enclosed by $a < \rho < b$ and $c < \rho$ is empty space. The outer wire carries the same current I in the opposite direction. Let the direction of z-axis be along the wire.

   (a) Use Ampere's law to find the expression for magnetic field in the four regions, $\rho < a$, $a < \rho < b$, $b < \rho < c$, and $c < \rho$.

   (b) Plot the resulting magnetic field as a function of ρ.

7. (**30 points.**) Consider a straight wire of radius a carrying current I described using the current density
   $$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} \frac{C}{\rho} e^{-\lambda\rho} \theta(a-\rho), \tag{10.23}$$
   where $\theta(x) = 1$ for $x > 1$ and zero otherwise.

   (a) Find C in terms of the current I.

   (b) Find the magnetic field inside and outside the wire.

   (c) Plot the magnetic field as a function of ρ.

8. (**30 points.**) A steady current I flows down a long cylindrical wire of radius a. The current density in the wire is described by, $n > 0$,
   $$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} \frac{I}{2\pi a^2} (n+2) \left(\frac{\rho}{a}\right)^n \theta(a-\rho). \tag{10.24}$$

(a) Show that, indeed,

$$\int_S d\mathbf{S}\cdot\mathbf{J}(\mathbf{r}) = I. \tag{10.25}$$

(b) Using Ampere's law show that the magnetic field inside and outside the cylinder is given by

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \frac{\mu_0}{4\pi}\frac{2I}{\rho}\left(\frac{\rho}{a}\right)^{n+2}\hat{\boldsymbol{\phi}} & \rho < a, \\ \frac{\mu_0}{4\pi}\frac{2I}{\rho}\hat{\boldsymbol{\phi}} & \rho > a. \end{cases} \tag{10.26}$$

(c) Plot the magnitude of the magnetic field as a function of ρ.

10.4 Magnetic vector potential

1. (**20 points.**) The magnetic field $\mathbf{B}(\mathbf{r})$ is given in terms of the magnetic vector potential $\mathbf{A}(\mathbf{r})$ by the relation

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}. \tag{10.27}$$

Find a magnetic vector potential (up to a gauge) for the uniform magnetic field

$$\mathbf{B} = B\,\hat{\mathbf{z}}. \tag{10.28}$$

Then, find another solution for $\mathbf{A}$ (up to a gauge) that is different from your original solution by more than just a constant. If you designed an experiment to measure $\mathbf{A}$, which one of your solution will the experiment measure?

2. (**20 points.**) A homogeneous magnetic field $\mathbf{B}$ is characterized by the vector potential

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}. \tag{10.29}$$

(a) Evaluate $\boldsymbol{\nabla}\times\mathbf{A}$.

(b) Verify that this construction satisfies the radiation gauge by showing that

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{10.30}$$

(c) Is this construction unique? No. Remember the freedom of gauge transformation,

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda(\mathbf{r}, t), \tag{10.31}$$

where $\lambda(\mathbf{r}, t)$ is an arbitrary function. Show that for any given vector potential $\mathbf{A}$ that satisfies the radiation gauge there exists λ that satisfies

$$-\nabla^2\lambda = 0 \tag{10.32}$$

that leads to the construction of $\mathbf{A}'$ satisfying the radiation gauge.

(d) Let us consider the special case when $\mathbf{B} = B\,\hat{\mathbf{z}}$.

i. Show that, for this case,

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} = -\frac{1}{2}By\,\hat{\mathbf{i}} + \frac{1}{2}Bx\,\hat{\mathbf{j}} = \frac{1}{2}B\rho\,\hat{\boldsymbol{\phi}}. \tag{10.33}$$

Visualize $\mathbf{A}$ diagramatically.

ii. Show that

$$\mathbf{A} = 0\,\hat{\mathbf{i}} + Bx\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{z}} \tag{10.34}$$

is a satisfactory vector potential for homogeneous magnetic field. Visualize $\mathbf{A}$ diagramatically. Show that this construction also satisfies the radiation gauge. Using

$$\lambda(\mathbf{r}, t) = \frac{1}{2} Bxy \tag{10.35}$$

construct $\mathbf{A}'$ that satisfies the radiation gauge. Evaluate $\boldsymbol{\nabla} \times \mathbf{A}'$.

iii. Show that

$$\mathbf{A} = -By\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{z}} \tag{10.36}$$

is also a satisfactory vector potential for homogeneous magnetic field. Visualize $\mathbf{A}$ diagramatically. Choose a suitable $\lambda(\mathbf{r}, t)$ to construct $\mathbf{A}'$ that also satisfies the radiation gauge. Evaluate $\boldsymbol{\nabla} \times \mathbf{A}'$.

iv. Chose an arbitray $\lambda(\mathbf{r}, t)$, of your choice, to construct another satisfactory vector potential for homogeneous magnetic field.

3. (**20 points.**) The magnetic field $\mathbf{B}$ is determined using the vector potential $\mathbf{A}$ by the relation

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{10.37}$$

Determine the vector potential for a uniform magnetic field pointing in the $\hat{\mathbf{z}}$ direction. Is this a unique construction.

4. (**20 points.**) Is it correct to conclude that

$$\boldsymbol{\nabla} \cdot (\mathbf{r} \times \mathbf{A}) = -\mathbf{r} \cdot (\boldsymbol{\nabla} \times \mathbf{A}), \tag{10.38}$$

where $\mathbf{A}$ is a vector dependent on $\mathbf{r}$? Explain your reasoning.

5. (**20 points.**) Is the relation

$$(\boldsymbol{\mu} \cdot \boldsymbol{\nabla}) \frac{\mathbf{r}}{r^3} = \boldsymbol{\nabla} \left(\frac{\boldsymbol{\mu} \cdot \mathbf{r}}{r^3} \right) \tag{10.39}$$

correct? ($\boldsymbol{\mu}$ is a position independent vector.)

(a) If yes, prove the relation.

(b) If not, disprove the relation.

6. (**20 points.**) Given the vector differential equation

$$\boldsymbol{\nabla} \phi(\mathbf{r}) = \frac{\mathbf{r} \times (\mathbf{a} \times \mathbf{r})}{r^3} \tag{10.40}$$

find $\phi(\mathbf{r})$ upto a constant. Here the vector $\mathbf{a}$ is uniform (constant) with respect to $\mathbf{r}$.

10.5 Straight wire

1. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics was found in terms of the vector potential $\mathbf{A}$ to be

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}. \tag{10.41}$$

(a) Verify that the above solution satisfies the Coulomb gauge condition, that is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0. \tag{10.42}$$

(b) Further, verify that the magnetic field is the curl of the vector potential and can be expressed in the form

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{10.43}$$

2. (**20 points.**) The solution to the Maxwell equations for the case of magnetostatics was found to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{10.44}$$

Verify that the above solution satisfies magnetostatics equations, that is, it satisfies

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{10.45}$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{10.46}$$

3. (**20 points.**) A steady current I flowing through an infinitely thin wire along the x-axis is described by the current density

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{x}}\, I\delta(z)\delta(y). \tag{10.47}$$

When the wire is along the y-axis it is described by the current density

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{y}}\, I\delta(z)\delta(x). \tag{10.48}$$

The above current densities satisfy

$$\int_S d\mathbf{a} \cdot \mathbf{J} = I, \tag{10.49}$$

where the integration is over an open surface S that crosses the wire once. Write the current density for a wire making an angle θ with respect to the x-axis and in the x-y plane.
Hint: Verify that the current density satisfies Eq. (10.49) for both the x-z plane and the y-z plane.

4. (**20 points.**) For an infinitely long wire of negligible thickness carrying a steady current I, described by

$$\mathbf{j}(\mathbf{r}) = \hat{\mathbf{z}} I\delta(x)\delta(y), \tag{10.50}$$

determine the magnetic field at an arbitrary point $\mathbf{r}$ using

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \tag{10.51}$$

5. (**50 points.**) Consider a wire segment of length $2L$ carrying a steady current I, described by

$$\mathbf{J}(\mathbf{r}) = \hat{\mathbf{z}} I\delta(x)\delta(y)\theta(-L<z<L), \tag{10.52}$$

when the rod is placed on the z-axis centered on the origin. Here $\theta(-L<z<L) = 0$, if $z > L$ and $z < -L$, and $\theta(-L<z<L) = 1$, otherwise.

(a) Show that the vector potential of the wire is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\, \hat{\mathbf{z}}\, I \left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right) \right]. \tag{10.53}$$

(b) Show that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}). \tag{10.54}$$

Also, using Eq. (10.54), verify that

$$\sinh^{-1}(-x) = -\sinh^{-1} x. \tag{10.55}$$

(c) Thus, express the vector potential of Eq. (10.53) in the form

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\,I\left[-2\ln\frac{\rho}{L} + F\left(\frac{z}{L},\frac{\rho}{L}\right)\right], \tag{10.56}$$

where $\rho^2 = x^2 + y^2$ and

$$F(a,b) = \ln[1 - a + \sqrt{(1-a)^2 + b^2}] + \ln[1 + a + \sqrt{(1+a)^2 + b^2}]. \tag{10.57}$$

(d) Show that

$$\mathbf{A}(\mathbf{r}) \xrightarrow{\rho\ll L, z\ll L} -\frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\,2I\ln\frac{\rho}{2L}. \tag{10.58}$$

(e) Using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ determine the magnetic field for an infinite rod (placed on the z-axis) to be

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{2I}{\rho}\hat{\boldsymbol{\phi}}. \tag{10.59}$$

6. (**50 points.**) (Based on Problem 5.8, Griffiths 4th edition.)
The magnetic field at position $\mathbf{r} = (x, y, z)$ due to a finite wire segment of length $2L$ carrying a steady current I, with the caveat that it is unrealistic (why?), placed on the z-axis with its end points at $(0, 0, L)$ and $(0, 0, -L)$, is

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{1}{\sqrt{x^2+y^2}}\left[\frac{z+L}{\sqrt{x^2+y^2+(z+L)^2}} - \frac{z-L}{\sqrt{x^2+y^2+(z-L)^2}}\right], \tag{10.60}$$

where $\hat{\boldsymbol{\phi}} = (-\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}) = (-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}})/\sqrt{x^2+y^2}$.

(a) Show that by taking the limit $L \to \infty$ we obtain the magnetic field near a long straight wire carrying a steady current I,

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}, \tag{10.61}$$

where $\rho = \sqrt{x^2+y^2}$ is the perpendicular distance from the wire.

(b) Show that the magnetic field on a line bisecting the wire segment is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}\frac{L}{\sqrt{\rho^2+L^2}}. \tag{10.62}$$

(c) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2+L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R}\frac{4}{\pi}\tan\frac{\pi}{4}. \tag{10.63}$$

(d) Show that the magnetic field at the center of a regular n-sided polygon, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R}\frac{n}{\pi}\tan\frac{\pi}{n}, \tag{10.64}$$

where R is the distance from center to a corner of the polygon.

(e) Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{10.65}$$

is obtained in the limit $n \to \infty$.

7. (**20 points.**) Find the magnetic field at the center of a square loop, which carries a steady current I. Let $2L$ be the length of a side, ρ be the distance from center to side, and $R = \sqrt{\rho^2 + L^2}$ be the distance from center to a corner. (Caution: Notation differs from Griffiths.) You should obtain

$$B = \frac{\mu_0 I}{2R} \frac{4}{\pi} \tan \frac{\pi}{4}. \tag{10.66}$$

Find the magnetic field at the center of a regular pentagon with the same R. Show that the magnetic field at the center of a regular n-sided polygon with same R, carrying a steady current I is

$$B = \frac{\mu_0 I}{2R} \frac{n}{\pi} \tan \frac{\pi}{n}, \tag{10.67}$$

where R is the distance from center to a corner of the polygon. Show that the magnetic field at the center of a circular loop of radius R,

$$B = \frac{\mu_0 I}{2R}, \tag{10.68}$$

is obtained in the limit $n \to \infty$.

8. (**20 points.**) The vector potential for a straight wire of infinite extent carrying a steady current I is

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi} \ln \frac{2L}{\rho}, \tag{10.69}$$

with $L \to \infty$ understood in the equation. The magnetic field around the wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi\rho}. \tag{10.70}$$

   (a) I leave the derivation of the above vector potential as an optional exercise, with bonus points worth **50 points** that could be used towards another homework.
   (b) Using an appropriate diagram describe the above vector potential and the magnetic field.
   (c) Evaluate $\boldsymbol{\nabla} \times \mathbf{A}$.

9. (**20 points.**) The vector potential for a straight wire of infinite extent carrying a steady current I is

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \hat{\mathbf{z}}\, 2I \ln \frac{\rho}{2L}, \qquad \rho \ll L, z \ll L, \tag{10.71}$$

where L is understood to be sufficiently larger than ρ and z (or $L \to \infty$) in the equation. Note that the restriction $\rho \ll L$, and $z \ll L$, is required to be consistent with $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$. The magnetic field around the wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi\rho}. \tag{10.72}$$

Starting from

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{10.73}$$

derive the relation for the flux of magnetic field

$$\Phi = \int_S d\mathbf{a} \cdot \mathbf{B} = \oint d\mathbf{l} \cdot \mathbf{A}. \tag{10.74}$$

Consider the loop to constitute a rectangle in the constant ϕ plane with $\rho_1 < \rho < \rho_2 < \infty$ and $-\infty < z_1 < z < z_2 < \infty$. Show that

$$\Phi = \frac{\mu_0}{4\pi} 2Ih \ln \frac{\rho_2}{\rho_1}, \tag{10.75}$$

where $h = z_2 - z_1$. What is the implication of the observation that the surface enclosing a closed curve is not unique. (Do not extend the surface to infinity to remain consistent with $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$.)

10. (**20 points.**) The magnetic field for a straight wire of infinite extent carrying a steady current I is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{2\pi\rho}. \tag{10.76}$$

Verify that

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0 \tag{10.77}$$

everywhere. In particular, investigate if the magnetic field is divergenceless on the wire, where $\rho = 0$. Next, evaluate

$$\boldsymbol{\nabla}\times\mathbf{B} \tag{10.78}$$

everywhere. Thus, check if the magnetic field due to a straight current carrying wire satisfies the two Maxwell equations relevant for magnetostatics.

10.6 Point magnetic dipole moment

1. (**20 points.**) Magnets are described by their magnetic moment. Estimate the magnetic moment $\mathbf{m}$ of Earth, assuming it to be a point magnetic dipole. Assume the magnitude of the Earth's magnetic field on its surface at the North pole to be $0.7\times 10^{-4}\,\text{T} = 0.7\,\text{Gauss}$. Next, similarly, estimate the magnetic moment of a typical refrigerator magnet.

2. (**20 points.**) A typical bar magnet is suitably approximated as a point magnetic dipole moment $\mathbf{m}$. The vector potential for a point magnetic dipole moment is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3}. \tag{10.79}$$

The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{10.80}$$

Consider the case when the point dipole is positioned at the origin and is pointing in the z-direction, i.e., $\mathbf{m} = m\,\hat{\mathbf{z}}$.

(a) Qualitatively plot the magnetic field lines for the dipole $\mathbf{m}$. (Hint: You do not have to depend on Eq. (10.80) for this purpose. An intuitive knowledge of magnetic field lines should be the guide.)

(b) Find the expression for the magnetic field on the negative z-axis. (Hint: On the negative z-axis we have, $\hat{\mathbf{r}} = -\hat{\mathbf{z}}$ and $r = z$.)

3. (**20 points.**) The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the point magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{10.81}$$

Let there be two point magnetic dipoles of equal strength. The first is positioned at the origin, $\mathbf{r}_1 = 0\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{k}}$, and is pointing in the $-\hat{\mathbf{x}}$ direction, $\mathbf{m}_1 = -m\,\hat{\mathbf{x}}$. The second is positioned on the z axis a distance $2a$ from the origin, $\mathbf{r}_2 = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + 2a\,\hat{\mathbf{k}}$, and is pointing in the $\hat{\mathbf{z}}$ direction, $\mathbf{m}_2 = m\,\hat{\mathbf{z}}$.

(a) Find the magnitude and direction of the magnetic field at the position $\mathbf{r} = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + a\,\hat{\mathbf{k}}$,

(b) Qualitatively plot the magnetic field lines in regions very far ($a \ll r$) from the dipoles.

4. (**20 points.**) The vector potential for a point magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{10.82}$$

Verify that the magnetic field due to the point dipole obtained by evaluating the curl

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{10.83}$$

can be expressed in the form, (using $\boldsymbol{\nabla}(1/r) = -\mathbf{r}/r^3$,)

$$\mathbf{B}(\mathbf{r}) = \mathbf{m}\, \mu_0\, \delta^{(3)}(\mathbf{r}) + \frac{\mu_0}{4\pi} (\mathbf{m} \cdot \boldsymbol{\nabla}) \left(\boldsymbol{\nabla} \frac{1}{r} \right). \tag{10.84}$$

In this form it is easier to verify that the magnetic field satisfies the Maxwell equation

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0. \tag{10.85}$$

Further, show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3} + \mathbf{m}\, \mu_0\, \delta^{(3)}(\mathbf{r}). \tag{10.86}$$

This form, for regions outside the point dipole, brings out the dipole field,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{10.87}$$

5. (**20 points.**) The vector potential for a point magnetic dipole moment is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{10.88}$$

The magnetic field due to a point magnetic dipole $\mathbf{m}$ at a distance $\mathbf{r}$ away from the magnetic dipole is given by the expression

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\big[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\big]}{r^3}, \qquad r \neq 0. \tag{10.89}$$

These expressions are for reference. This question, probably, can also be answered without relying on these expressions. Sketch the magnetic field lines due to two identical point magnetic dipole moments separated by a distance a and their dipole moments parallel to the line joining the two dipoles. Does the magnetic field go to zero anywhere? If yes, identify the point. If not, why not?

10.6.1 Rotating charged conductors

1. (**20 points.**) The vector potential for a point magnetic dipole moment $\mathbf{m}$ is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{10.90}$$

Determine the corresponding magnetic field due to the point dipole using

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{10.91}$$

Find the simplified expression for the magnetic field everywhere along the line collinear to the magnetic moment $\mathbf{m}$. Next, find the simplified expression for the magnetic field in the plane containing the magnetic moment and perpendicular to the magnetic moment $\mathbf{m}$.

2. (**40 points.**) (Based on Problem 5.58, Griffiths 4th edition.) A circular loop of wire carries charge q uniformly distributed on it. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(\rho - a)\delta(z - 0). \tag{10.92}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{10.93}$$

determine the magnetic dipole moment of this loop to be

$$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{10.94}$$

(c) Calculate the angular momentum of the rotating loop to be

$$\mathbf{L} = ma^2\boldsymbol{\omega}, \tag{10.95}$$

where m is the mass of the loop.

(d) What is the gyromagnetic ratio g of the rotating loop, which is defined by the relation $\mathbf{m} = g\mathbf{L}$.

3. (**40 points.**) A circular loop of wire carries charge q uniformly distributed on it. It rotates with angular velocity $\boldsymbol{\omega}$ about its axis, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{2\pi a}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(\rho - a)\delta(z - 0). \tag{10.96}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{10.97}$$

determine the magnetic dipole moment of this loop to be

$$\mathbf{m} = \frac{qa^2}{2}\boldsymbol{\omega}. \tag{10.98}$$

(c) Calculate the vector potential $\mathbf{A}(0, 0, z)$ on the z-axis.

(d) Calculate the magnetic field $\mathbf{B}(0, 0, z)$ on the z-axis.

4. (**20 points.**) A charged spherical shell of radius a has charge q uniformly distributed on it. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter, say z-axis.

(a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega} \times \mathbf{r}\,\delta(r - a). \tag{10.99}$$

Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ for circular motion.

(b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r} \times \mathbf{J}(\mathbf{r}). \tag{10.100}$$

determine the magnetic dipole moment of the rotating sphere to be

$$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{10.101}$$

5. (**40 points.**) A charged spherical shell of radius a has charge q uniformly distributed on it. It rotates with angular velocity $\boldsymbol{\omega}$ about a diameter.

   (a) Show that the current density generated by this motion is given by

$$\mathbf{J}(\mathbf{r}) = \frac{q}{4\pi a^2}\,\boldsymbol{\omega}\times\mathbf{r}\,\delta(r-a). \tag{10.102}$$

   Hint: Use $\mathbf{J}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$ and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ for circular motion.

   (b) Using

$$\mathbf{m} = \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{J}(\mathbf{r}). \tag{10.103}$$

   determine the magnetic dipole moment of the rotating sphere to be

$$\mathbf{m} = \frac{qa^2}{3}\boldsymbol{\omega}. \tag{10.104}$$

   (c) Evaluate the vector potential inside and outside the sphere to be

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{r}}{a^3}, & r < a, \\ \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{r}}{r^3}, & a < r. \end{cases} \tag{10.105}$$

   Hint: Out of the three vectors $\boldsymbol{\omega}$, the observation point $\mathbf{r}$, and the integration variable $\mathbf{r}'$, choose $\mathbf{r}$ to be along the z axis while working in spherical polar coordinates. This leads to considerable simplification in the expression for $|\mathbf{r}-\mathbf{r}'|$ appearing in the denominator. Otherwise, without choosing $\mathbf{r}$ to be along $\hat{\mathbf{z}}$, use the ideas of Legendre polynomials and spherical harmonics.

   (d) Derive the corresponding expression for the magnetic field, using $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$, to be

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{2\mathbf{m}}{a^3}, & r < a, \\ \dfrac{\mu_0}{4\pi}\dfrac{1}{r^3}\Big[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}\Big], & a < r. \end{cases} \tag{10.106}$$

6. (**100 points.**) The vector potential inside a rotating charged conducting shell is given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{Q}{3R}\,\boldsymbol{\omega}_0\times\mathbf{r}, \qquad r < R, \tag{10.107}$$

   where Q is the total charge on the conducting shell of radius R that is rotating with angular velocity $\boldsymbol{\omega}_0$.

   (a) Show that the magnetic field produced by this motion is given

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \frac{\mu_0}{4\pi}\frac{2Q}{3R}\,\boldsymbol{\omega}_0, \qquad r < R, \tag{10.108}$$

   which is uniform inside the shell.

   (b) A charged particle takes a circular path (in general a helical path). Describe the motion of an electron inside this rotating shell. In particular, calculate the expression for the angular speed of rotation ω of the electron.

   (c) Next, consider a current carrying loop of wire inside the shell. The interaction energy of this loop with the rotating shell is given by

$$W_m = -\int d^3r\,\mathbf{J}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r}), \tag{10.109}$$

where $\mathbf{J}(\mathbf{r})$ is the current density of the current carrying loop. Show that this interaction energy can be expressed in terms of the magnetic field as

$$W_m = -I \int_S d\mathbf{S} \cdot \mathbf{B}(\mathbf{r}) = -I \mathbf{\Phi}_m, \tag{10.110}$$

where I is the current in the loop, S represents any surface bounded by the loop, and $\mathbf{\Phi}_m$ is the magnetic flux through the loop.

(d) Calculate the interaction energy between a circular loop of wire of radius a carrying a current I with the symmetrical axis of the loop along the direction $\mathbf{n}$.

(e) Torque is defined as negative change in energy with respect to a change in rotation angle θ,

$$\tau = -\frac{\partial}{\partial \theta} W_m. \tag{10.111}$$

Define $\cos\theta = \mathbf{n} \cdot \hat{\boldsymbol{\omega}}_0$ and calculate the torque on the current loop inside the shell.

(f) Precession of a spinning top is understood in terms of the torque equation. Do you expect a current loop inside the rotating shell to precess? Or, in general, a particle with magnetic moment to precess inside the rotating shell?

10.7 Dirac string: Infinitely thin solenoid

1. (**20 points.**) It is a bit perplexing that the magnetic field due to an infinitely long solenoid is independent of the radius of the solenoid. It prompts us to investigate the limiting case when the radius of the solenoid goes to zero. Such a solenoid can be imagined to be built out of point magnetic moments stacked up along the z axis with all their moments pointing along the z axis. Let us define the magnetic moment per unit length for a such an infinitely thin 'solenoid' to be

$$\frac{\text{magnetic moment}}{\text{length}} = \boldsymbol{\lambda} = \hat{\mathbf{z}} \frac{d\mathbf{m}}{dz}. \tag{10.112}$$

(a) The magnetic vector potential for an infinitesimal element of the 'solenoid' is that of a point magnetic dipole given by

$$d\mathbf{A} = \frac{\mu_0}{4\pi} \frac{d\mathbf{m} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \frac{\lambda \rho \, dz'}{[\rho^2 + (z - z')^2]^{\frac{3}{2}}}. \tag{10.113}$$

Thus, the total magnetic vector potential at any point is obtained by the vector sum of all the elements, using integration,

$$\mathbf{A}(\mathbf{r}) = \int d\mathbf{A} = \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \int_{-\infty}^{\infty} \frac{\lambda \rho \, dz'}{[\rho^2 + (z - z')^2]^{\frac{3}{2}}}. \tag{10.114}$$

Complete the integral and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \frac{2\lambda}{\rho}. \tag{10.115}$$

(b) Evaluate the magnetic field for this magnetic vector potential using the relation

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{10.116}$$

In particular, show that

$$\mathbf{B} = 0 \quad \text{for} \quad \rho \neq 0. \tag{10.117}$$

(c) Using Stoke's theorem deduce

$$\int_S d\mathbf{S}\cdot\mathbf{B} = \oint_S d\mathbf{l}\cdot\mathbf{A}, \tag{10.118}$$

where S is a surface. Let S be a planar disc perpendicular to the z axis and centered on the axis. Is there an inconsistency? Show that the inconsistency is avoided by having the magnetic field to be

$$\mathbf{B} = \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi}(4\pi\lambda)\delta^{(2)}(\boldsymbol{\rho}). \tag{10.119}$$

(d) Evaluate

$$\boldsymbol{\nabla}\cdot\mathbf{B}. \tag{10.120}$$

Is it zero?

2. (**20 points.**) It is a bit perplexing that the magnetic field due to an infinitely long solenoid is independent of the radius of the solenoid. It prompts us to investigate the limiting case when the radius of the solenoid goes to zero. Such a solenoid can be imagined to be built out of point magnetic moments stacked up along the z axis with all their moments pointing along the z axis. Let us define the magnetic moment per unit length for a such an infinitely thin 'solenoid' to be

$$\frac{\text{magnetic moment}}{\text{length}} = \boldsymbol{\lambda} = \hat{\mathbf{z}}\frac{d\mathbf{m}}{dz}. \tag{10.121}$$

A Dirac string constitutes of an infinitely thin solenoid that extends from a point $\mathbf{r}'$ to infinity along an arbitrary curve. Let us consider a Dirac string that extends from the origin to infinity along a straight line, the negative z axis. Thus, we can write the magnetic moment per unit volume

$$\hat{\mathbf{z}}\,\lambda\delta(x)\delta(y). \tag{10.122}$$

(a) The magnetic vector potential for an infinitesimal element of the 'solenoid' is that of a point magnetic dipole given by

$$d\mathbf{A} = \frac{\mu_0}{4\pi}\frac{d\mathbf{m}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{\lambda\rho\,dz'}{[\rho^2+(z-z')^2]^{\frac{3}{2}}}. \tag{10.123}$$

Thus, the total magnetic vector potential at any point is obtained by the vector sum of all the elements, using integration,

$$\mathbf{A}(\mathbf{r}) = \int d\mathbf{A} = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\int_{-\infty}^{0}\frac{\lambda\rho\,dz'}{[\rho^2+(z-z')^2]^{\frac{3}{2}}}. \tag{10.124}$$

Complete the integral and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{\lambda}{r}\frac{(1-\cos\theta)}{\sin\theta}. \tag{10.125}$$

(b) Evaluate the magnetic field for this magnetic vector potential using the relation

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \tag{10.126}$$

and show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\lambda\frac{\hat{\mathbf{r}}}{r^2}. \tag{10.127}$$

Compare this to the magnetic field due to a magnetic monopole.

(c) Evaluate

$$\boldsymbol{\nabla}\cdot\mathbf{B}. \tag{10.128}$$

Is it zero?

(d) Use Eq. (10.86) to answer associated questions in Problem 6.18 in Jackson.

10.8 Multipole expansion

1. (**20 points.**) Determine the total magnetic dipole moment for the following configuration. The current in the loop is I and each fold in the loop is of length a.

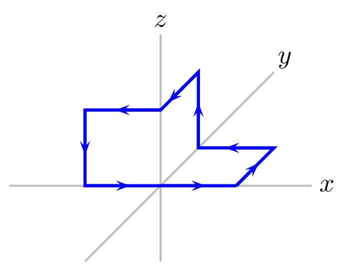

Figure 10.4: Problem 1

2. (**20 points.**) Determine the total magnetic dipole moment for the following configuration. The current in the loop is I and each fold in the loop is of length a.

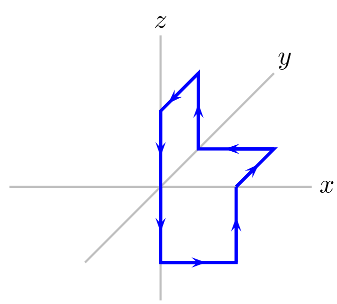

Figure 10.5: Problem 2

3. (**20 points.**) Find the total magnetic dipole moment of a helical coil carrying current I and n turns per unit length, bent in the shape of torus of major radius R and minor radius a. Assume the coil to be tightly wound so that each current loop can be assumed to be a circle of radius a.
   Solution: Zero.

4. (**20 points.**) Find the total magnetic dipole moment of a helical coil carrying current I and n turns per unit length, bent in the shape of half-torus of major radius R and minor radius a. Assume the coil to be tightly wound so that each current loop can be assumed to be a circle of radius a.

10.9 Circular loop of wire

10.9.1 On the axis

1. (**20 points.**) A circular wire carrying current I forms a loop of radius a and is described by current density

$$\mathbf{j}(\mathbf{r}') = \hat{\boldsymbol{\phi}}' I \delta(z') \delta(\rho' - a). \tag{10.129}$$

Determine the magnetic vector potential using

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \tag{10.130}$$

on the axis of the circular wire at $\mathbf{r} = z\,\hat{\mathbf{k}}$. Determine the magnetic field using

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\mathbf{j}(\mathbf{r}') \times \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{10.131}$$

on the axis of the circular wire at $\mathbf{r} = z\,\hat{\mathbf{k}}$.

10.9.2 Complete elliptic integrals

Complete elliptic integrals of the first and second kind can be defined using the integral representations,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1-k^2\sin^2\psi}}, \tag{10.132a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1-k^2\sin^2\psi}, \tag{10.132b}$$

respectively.

1. (**20 points.**) Verify that

$$K(0) = \frac{\pi}{2}, \tag{10.133a}$$

$$E(0) = \frac{\pi}{2}. \tag{10.133b}$$

Then, verify that

$$E(1) = 1. \tag{10.134}$$

Note that

$$K(1) = \int_0^{\frac{\pi}{2}} \frac{d\psi}{\cos\psi} \tag{10.135}$$

is divergent. To see the nature of divergence we introduce a cutoff parameter $\delta > 0$ and write

$$K(1) = \int_0^{\frac{\pi}{2}-\delta} \frac{d\psi}{\cos\psi}. \tag{10.136}$$

Evaluate the integral, (using the identity $d(\sec\psi + \tan\psi)/d\psi = \sec\psi(\sec\psi + \tan\psi)$,) and show that

$$K(1) \sim \ln 2 - \ln\delta - \frac{\delta^2}{12} + \mathcal{O}(\delta)^4 \tag{10.137}$$

has logarithmic divergence. Using Mathematica (or another graphing tool) plot $K(k)$ and $E(k)$ as functions of k for $0 \leq k < 1$.

2. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \ldots\right], \tag{10.138a}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{(1-2n)} = \frac{\pi}{2}\left[1 - \frac{1}{4}k^2 - \frac{3}{64}k^4 - \ldots\right]. \tag{10.138b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the next-to-leading order contributions in the above series expansions by expanding the radical in Eqs.(10.132) as a series. Use

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \ldots, \tag{10.139a}$$

$$\sqrt{1-x} = 1 - \frac{1}{2}x + \ldots. \tag{10.139b}$$

3. (**20 points.**) The complete elliptic integrals have the power series expansions

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1+\frac{1}{4}k^2+\frac{9}{64}k^4+\ldots\right], \tag{10.140a}$$

$$E(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{k^{2n}}{(1-2n)} = \frac{\pi}{2}\left[1-\frac{1}{4}k^2-\frac{3}{64}k^4-\ldots\right]. \tag{10.140b}$$

The leading order contribution in the power series expansions are from $K(0)$ and $E(0)$. Evaluate the leading order contribution of

$$\left[K(k) - \frac{(2-k^2)}{2(1-k^2)}E(k)\right]. \tag{10.141}$$

Hint: Truncate all series expansions to order k^0 and collect the terms. If it is zero, repeat for order k^2. Repeat for subsequent higher orders until you obtain a non-zero contribution.

4. (**20 points.**) Show that the perimeter of an ellipse, characterized by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{10.142}$$

with eccentricity

$$e=\sqrt{1-\frac{b^2}{a^2}}, \tag{10.143}$$

is given by

$$C = 4aE(e), \tag{10.144}$$

where $E(k)$ is the complete elliptic integral of the second kind,

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi\sqrt{1-k^2\sin^2\psi}. \tag{10.145}$$

A circle is an ellipse with zero eccentricity. Deduce the circumference of a circle using the formula.

5. (**20 points.**) Refer problem on pendulum in Classical Mechanics Notes. This introduces the elliptic integral of second kind.

6. (**20 points.**) Show that

$$\frac{d}{dk}E(k) = \frac{1}{k}\big[E(k)-K(k)\big]. \tag{10.146}$$

Show that

$$\frac{d}{dk}K(k) = \frac{1}{k}\big[\pi(k)-K(k)\big], \tag{10.147}$$

where

$$\pi(k) = \int_0^{\frac{\pi}{2}} d\psi\frac{1}{(1-k^2\sin^2\psi)^{\frac{3}{2}}}. \tag{10.148}$$

Show that $\pi(k)$ can be expressed in term of the complete elliptic integrals as

$$\pi(k) = \frac{E(k)}{(1-k^2)}. \tag{10.149}$$

7. (**20 points.**) Evaluate

$$\frac{d^2}{dk^2}E(k), \quad \frac{d^2}{dk^2}K(k) \tag{10.150}$$

10.9.3 Exact result in terms of elliptic integrals

1. (**30 points.**) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I\delta(\rho - a)\delta(z), \tag{10.151}$$

where the the loop is chosen to be in the x-y plane with the origin as its center.

(a) Verify that

$$\int_S d\mathbf{a}\cdot\mathbf{j} = I, \tag{10.152}$$

where surface S is a half-plane of constant ϕ.

(b) Show that magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\hat{\boldsymbol{\phi}}'}{\sqrt{z^2+\rho^2+a^2-2\rho a\cos(\phi-\phi')}}. \tag{10.153}$$

(c) Substitute $\phi' - \phi \to \phi'$ and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi} a \int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{z^2+\rho^2+a^2-2\rho a\cos\phi'}}. \tag{10.154}$$

(d) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1-k^2\sin^2\psi}}, \tag{10.155a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1-k^2\sin^2\psi}, \tag{10.155b}$$

respectivly, the magnetic vector potential is

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{4a}{\sqrt{z^2+(\rho+a)^2}}\left[\frac{2}{k^2}\Big\{K(k)-E(k)\Big\} - K(k)\right], \tag{10.156}$$

where

$$k^2 = \frac{4a\rho}{z^2+(\rho+a)^2}. \tag{10.157}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (10.154) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2+\rho^2+a^2+2\rho a\cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals.

2. (**30 points.**) We have earlier found the magnetic vector potential to be zero everywhere along the symmetry axis of the circular loop. With our exact expression let us calculate an approximate expression for the magnetic vector potential very close to the axis. Using the power series expansions for the complete elliptic integrals show that

$$\frac{2}{k^2}\Big\{K(k)-E(k)\Big\} - K(k) = \frac{\pi}{16}k^2 + \ldots. \tag{10.158}$$

Drop the next-to-leading order terms, valid when $k \ll 1$, and show that

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, A(\rho,z) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0 I}{4\pi}\frac{a^2\pi\rho}{[z^2+(\rho+a)^2]^{\frac{3}{2}}}. \tag{10.159}$$

Check that $\mathbf{A} = 0$ on the axis. Show that the magnetic field close to the axis ($k \ll 1$) is given by

$$\mathbf{B}(\mathbf{r}) = -\hat{\boldsymbol{\rho}}\frac{\partial A}{\partial z} + \hat{\mathbf{z}}\left(\frac{\partial}{\partial\rho}+\frac{1}{\rho}\right)A. \tag{10.160}$$

3. (**20 points.**) The expression for the magnetic vector potential **A** and the magnetic field **B** for a circular loop of radius a carrying a current I is given in terms of the complete elliptic integrals. An approximate expression for the magnetic vector potential close to the axis is

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, A(\rho, z) = \hat{\boldsymbol{\phi}}\, \frac{\mu_0 I}{4\pi} \frac{a^2 \pi \rho}{[z^2 + (\rho + a)^2]^{\frac{3}{2}}}. \tag{10.161}$$

Check that $\mathbf{A} = 0$ on the axis. The magnetic field close to the axis, then, is calculated using

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{10.162}$$

Show that the magnetic field close to the axis ($k \ll 1$) is given by

$$\mathbf{B}(\mathbf{r}) = -\hat{\boldsymbol{\rho}} \frac{\partial A}{\partial z} + \hat{\mathbf{z}} \left(\frac{\partial A}{\partial \rho} + C \right). \tag{10.163}$$

Find C.

4. (**30 points.**) The current density for a circular loop of radius a carrying a steady current I is given by

$$\mathbf{j}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\, I \delta(\rho - a) \delta(z), \tag{10.164}$$

where the the loop is chosen to be in the x-y plane with the origin as its center.

(a) Using Bio-Savart law and completing the integrals involving δ-functions show that magnetic field has the form

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\phi' \frac{\left[a^2 \hat{\mathbf{z}} + az \hat{\boldsymbol{\rho}}' - a\rho (\hat{\boldsymbol{\rho}} \times \hat{\boldsymbol{\phi}}') \right]}{[z^2 + \rho^2 + a^2 - 2\rho a \cos(\phi - \phi')]^{\frac{3}{2}}}. \tag{10.165}$$

(b) Substitute $\phi' - \phi \to \phi'$ and show that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} d\phi' \frac{\left[(a^2 - a\rho \cos\phi') \hat{\mathbf{z}} + az \hat{\boldsymbol{\rho}} \cos\phi' + az \hat{\boldsymbol{\phi}} \sin\phi' \right]}{[z^2 + \rho^2 + a^2 - 2\rho a \cos\phi']^{\frac{3}{2}}}. \tag{10.166}$$

(c) The ϕ' integral can not be completed in terms of elementary functions. Show that in terms of the complete elliptic integrals of the first and second kind,

$$K(k) = \int_0^{\frac{\pi}{2}} d\psi \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}}, \tag{10.167a}$$

$$E(k) = \int_0^{\frac{\pi}{2}} d\psi \sqrt{1 - k^2 \sin^2 \psi}, \tag{10.167b}$$

respectivly, the magnetic field is

$$\begin{aligned} \mathbf{B}(\mathbf{r}) = {} & \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \left[K(k) - \frac{(z^2 + \rho^2 - a^2)}{z^2 + (\rho - a)^2} E(k) \right] \\ & - \hat{\boldsymbol{\rho}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \frac{z}{\rho} \left[K(k) - \frac{(z^2 + \rho^2 + a^2)}{z^2 + (\rho - a)^2} E(k) \right], \end{aligned} \tag{10.168}$$

where

$$k^2 = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{10.169}$$

Hint: Show that the contributions to the ϕ' integral in Eq. (10.154) gets equal contributions from 0 to π and π to 2π. In particular, use the form with $(z^2 + \rho^2 + a^2 + 2\rho a \cos\phi')$ in the denominator. Then, use the half-angle formula to obtain the integral in terms of the complete elliptic integrals. It is useful to identify

$$\int_0^{\frac{\pi}{2}} d\psi \frac{1}{(1 - k^2 \sin^2 \psi)^{\frac{3}{2}}} = \frac{E(k)}{(1 - k^2)}. \tag{10.170}$$

5. (**20 points.**) A circular loop of radius a carrying a steady current I with the loop chosen to be in the x-y plane with the origin at the center of the loop has the the magnetic vector potential given by

$$\mathbf{A}(\mathbf{r}) = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{4\pi} \frac{4a}{\sqrt{z^2 + (\rho + a)^2}} \left[\frac{2}{k^2} \Big\{ K(k) - E(k) \Big\} - K(k) \right], \tag{10.171}$$

where

$$k^2 = \frac{4a\rho}{z^2 + (\rho + a)^2}. \tag{10.172}$$

The magnetic field is

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \hat{\mathbf{z}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \left[K(k) - \frac{(z^2 + \rho^2 - a^2)}{z^2 + (\rho - a)^2} E(k) \right] \\ &\quad - \hat{\boldsymbol{\rho}} \frac{\mu_0 I}{4\pi} \frac{2}{\sqrt{z^2 + (\rho + a)^2}} \frac{z}{\rho} \left[K(k) - \frac{(z^2 + \rho^2 + a^2)}{z^2 + (\rho - a)^2} E(k) \right]. \end{aligned} \tag{10.173}$$

We can evaluate the vector potential and the magnetic field close to the symmetry axis of the loop using the approximation $k^2 \ll 1$ in the above expressions. Using

$$\frac{(z^2 + \rho^2 + a^2)}{z^2 + (\rho - a)^2} = \frac{(2 - k^2)}{2(1 - k^2)} = 1 + \frac{k^2}{2} + \frac{k^4}{2} + \ldots, \tag{10.174a}$$

$$\begin{aligned} \frac{(z^2 + \rho^2 - a^2)}{z^2 + (\rho - a)^2} &= \frac{(2 - k^2)}{2(1 - k^2)} - \frac{a}{2\rho} \frac{k^2}{(1 - k^2)} \\ &= \left[1 + \frac{k^2}{2} + \frac{k^4}{2} + \ldots \right] - \frac{a}{2\rho} \left[k^2 + k^4 + \ldots \right] \end{aligned} \tag{10.174b}$$

we can show that

$$\begin{aligned} \frac{2}{k^2} \Big\{ K(k) - E(k) \Big\} - K(k) &= \frac{\pi}{16} k^2 + \frac{3\pi}{64} k^4 + \ldots \\ &= \frac{(\pi a^2)}{[z^2 + (\rho + a)^2]} \frac{\rho}{4a} \left[1 + \frac{3a\rho}{[z^2 + (\rho + a)^2]} + \ldots \right], \end{aligned} \tag{10.175a}$$

$$\begin{aligned} K(k) - \frac{(z^2 + \rho^2 - a^2)}{z^2 + (\rho - a)^2} E(k) &= \left[-\frac{3\pi}{32} k^4 + \ldots \right] + \frac{\pi a}{4\rho} \left[k^2 + \frac{3}{4} k^4 + \ldots \right] \\ &= \frac{(\pi a^2)}{[z^2 + (\rho + a)^2]} \left[1 - \frac{3}{2} \frac{\rho(\rho - 2a)}{[z^2 + (\rho + a)^2]} + \ldots \right], \end{aligned} \tag{10.175b}$$

$$\begin{aligned} K(k) - \frac{(z^2 + \rho^2 + a^2)}{z^2 + (\rho - a)^2} E(k) &= -\frac{3\pi}{32} k^4 + \ldots \\ &= -\frac{3}{2} \frac{(\pi a^2) \rho^2}{[z^2 + (\rho + a)^2]^2} + \ldots. \end{aligned} \tag{10.175c}$$

Using these approximations, which are appropriate for regions close to the axis ($k^2 \ll 1$,) we have

$$\mathbf{A}(\mathbf{r}) \xrightarrow{k^2 \ll 1} \hat{\boldsymbol{\phi}} \frac{\mu_0}{4\pi} \frac{I(\pi a^2)\rho}{[z^2 + (\rho + a)^2]^{\frac{3}{3}}} \left[1 + \frac{3a\rho}{[z^2 + (\rho + a)^2]} \right] \tag{10.176}$$

and

$$\mathbf{B}(\mathbf{r}) \xrightarrow{k^2 \ll 1} \hat{\mathbf{z}} \frac{\mu_0}{4\pi} \frac{I(\pi a^2) 2}{[z^2 + (\rho + a)^2]^{\frac{3}{2}}} \left[1 - \frac{3}{2} \frac{\rho(\rho - 2a)}{[z^2 + (\rho + a)^2]} \right] - \hat{\boldsymbol{\rho}} \frac{\mu_0}{4\pi} \frac{I(\pi a^2) 3\rho z}{[z^2 + (\rho + a)^2]^{\frac{5}{2}}}. \tag{10.177}$$

10.10 Solenoid

1. (**10 points.**) The magnetic field inside a solenoid of radius a, and of infinite extension in the direction of the axis, is given by the expression

$$\mathbf{B}(\mathbf{r}) = \hat{\mathbf{n}}\mu_0 I n, \tag{10.178}$$

where n is the number of turns per unit length, I is the current, and $\hat{\mathbf{n}}$ points along the axis determined by the cross product of direction of radius vector and direction of current.

   (a) If you double the radius of the solenoid, how much does the magnetic field inside the solenoid change?

   (b) The force on a charge particle due to a magnetic field is given by $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$. What is the force experienced by a charge particle q cruising on the axis of the solenoid with speed v?

2. (**30 points.**) The current density for a wire forming a helix and carrying a steady current I is given by

$$\mathbf{J}(\mathbf{r}) = \mathbf{n}\, I \sum_{m=-\infty}^{\infty} \frac{1}{\rho}\delta(\rho - a)\delta\left(\phi - 2\pi\frac{z}{L} + 2\pi(m-1)\right), \tag{10.179}$$

where the direction of the flow of current is described by the vector

$$\mathbf{n} = \hat{\mathbf{z}} + 2\pi\frac{a}{L}\,\hat{\boldsymbol{\phi}}. \tag{10.180}$$

Here a is the radius of the helix and L is the pitch. The coordinates (ρ, ϕ, z) are the usual cylindrical coordinates. The coordinate ϕ generates one period of the helix for $0 < \phi < 2\pi$, and the sum periodically repeats it.

   (a) Calculate the flux of current density

$$\int_S d\mathbf{a} \cdot \mathbf{J}(\mathbf{r}) \tag{10.181}$$

   passing through the surface S of the $z = 0$ plane.
   Hint: The area element for the $z = 0$ plane is $d\mathbf{a} = \hat{\mathbf{z}}\, dx\, dy = \hat{\mathbf{z}}\, \rho d\rho\, d\phi$.

   (b) Using

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{J}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \tag{10.182}$$

   it is possible to determine the magnetic field on the symmetry axis of the helix in terms of the modified Bessel functions,

$$\mathbf{B}(0, 0, z) = \frac{\mu_0 I}{L}\left[\hat{\mathbf{z}} - \hat{\boldsymbol{\phi}}\left\{\frac{2\pi a}{L}K_0\left(\frac{2\pi a}{L}\right) + K_1\left(\frac{2\pi a}{L}\right)\right\}\right]. \tag{10.183}$$

   How is the $\hat{\mathbf{z}}$-component of the magnetic field related to the magnetic field of a solenoid?

   (c) Using the fact that the modified Bessel functions for large arguments tends to zero determine the magnetic field on the z-axis in this limit. How well does a helix with pitch L small compared to radius a compare with a solenoid?

10.11 Magnetostatic energy

1. (**20 points.**) The current density for a straight wire of infinite extent carrying a steady current I_1 in the $\hat{\mathbf{z}}$ direction and passing through the origin is

$$\mathbf{j}_1(\mathbf{r}) = \hat{\mathbf{z}} I_1 \delta(x)\delta(y). \tag{10.184}$$

The magnetic vector potential generated by the stright wire is given by

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi} 2I_1 \ln\frac{2L}{\rho}, \tag{10.185}$$

where $\rho = \sqrt{x^2 + y^2}$ and $L \to \infty$ is understood in the equation. The magnetic field around the straight wire is given by

$$\mathbf{B}(\mathbf{r}) = \hat{\boldsymbol{\phi}}\,\frac{\mu_0}{4\pi}\frac{2I_1}{\rho}. \tag{10.186}$$

Let there be another straight wire of infinite extent carrying a steady current I_2 in the $\hat{\mathbf{z}}$ direction and passing through $x = a$ and $y = 0$ such that the two wires are parallel with separation distance a. The current density for the second wire is

$$\mathbf{j}_2(\mathbf{r}) = \hat{\mathbf{z}} I_2 \delta(x - a)\delta(y). \tag{10.187}$$

(a) The magnetostatic interaction energy of such a configuration of two wires is given by

$$W_{12} = -\int d^3r\,\mathbf{A}_1(\mathbf{r}) \cdot \mathbf{j}_2(\mathbf{r}), \tag{10.188}$$

where 1 in the subscript of $\mathbf{A}_1$ signifies that it is the magnetic vector potential due to the first wire. Similarly, $\mathbf{j}_2$ is the current density of the second wire. Find the expression for the interaction energy per unit length for the configuration of the two parallel wires to be

$$\frac{W_{12}}{2L} = -\frac{\mu_0}{4\pi} 2I_1 I_2 \ln\frac{2L}{a}. \tag{10.189}$$

Verify that the above expression for the interaction energy per unit length is consistent with the experimental observation that 'like' currents attract and 'unlike' currents repel.

(b) The magnetostatic energy for a configuration of two wires is

$$W_m = -\frac{1}{2}\frac{\mu_0}{4\pi}\int d^3r \int d^3r'\,\frac{\mathbf{j}(\mathbf{r}) \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}, \tag{10.190}$$

where

$$\mathbf{j}(\mathbf{r}) = \mathbf{j}_1(\mathbf{r}) + \mathbf{j}_2(\mathbf{r}). \tag{10.191}$$

Observe the breakup of energy into

$$W_m = W_1 + W_2 + W_{12}, \tag{10.192}$$

where W_1 and W_2 are the self energies of the individual wires and W_{12} is the interaction energy. Note the role of the factor of half in counting the interactions. Show that

$$\frac{W_{12}}{2L} = -\frac{\mu_0}{4\pi} 2I_1 I_2 \left[\ln\frac{2L}{a} + \ln 2 - 1\right], \tag{10.193}$$

which is consistent upto a constant.

(c) Evaluate

$$\mathbf{F} = -\boldsymbol{\nabla}_a W_m. \tag{10.194}$$

(d) Evaluate

$$\mathbf{F}_{12} = -\int d^3r\,\mathbf{j}_1(\mathbf{r}) \times \mathbf{B}_2(\mathbf{r}). \tag{10.195}$$

Chapter 11

Action for electromagnetism

11.1 Action

1. (**20 points.**) Introduce the Poisson bracket and introduce the Hamiltonian for a charge particle in a magnetic field.

   Refer the problem under Canonical transformations in Note on Calssical Mechanics.

Chapter 12

Special Relativity

12.1 Relativity principle

Problems

1. (**20 points.**) The relativity principle states that the laws of physics are invariant (or covariant) when observed using different coordinate systems. In special relativity we restrict these coordinate systems to be uniformly moving with respect to each other. Let $z = z' = 0$ at $t = 0$.

   (a) Linear: Spatial homogeneity, spatial isotropy, and temporal homogeneity, require the transformation to be linear. (We will skip this derivation.) Then, for simplicity, restricting to coordinate systems moving with respect to each other in a single direction, we can write

   $$z' = A(v)\, z + B(v)\, t, \tag{12.1a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{12.1b}$$

   We will refer to the respective frames as primed and unprimed.

   (b) Identity: An object P at rest in the primed frame, described by $z' = 0$, will be described in the unprimed frame as $z = vt$.

   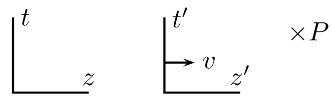

   Figure 12.1: Identity.

   Using these in Eq. (12.1a), we have

   $$0 = A(v)\, vt + B(v)\, t. \tag{12.2}$$

   This implies $B(v) = -vA(v)$. Thus, show that

   $$z' = A(v)\,(z - vt), \tag{12.3a}$$
   $$t' = E(v)\, z + F(v)\, t. \tag{12.3b}$$

   (c) Reversal: The descriptions of a process in the unprimed frame moving to the right with velocity v with respect to the primed should be identical to those made in the unprimed (with their axis flipped)

Figure 12.2: Reversal.

moving with velocity $-v$ with respect to the primed (with their axis flipped). This is equivalent to the requirement of isotropy in an one dimensional space.
That is, the transformation must be invariant under

$$z \to -z, \qquad z' \to -z', \qquad v \to -v. \tag{12.4}$$

This implies

$$\begin{aligned} -z' &= A(-v)\,(-z + vt), &(12.5a)\\ t' &= -E(-v)\,z + F(-v)\,t. &(12.5b)\end{aligned}$$

Show that Eqs. (12.3a) and (12.5a) in conjunction imply

$$A(-v) = A(v). \tag{12.6}$$

Further, show that Eqs. (12.3b) and (12.5b) in conjunction implies

$$\begin{aligned} E(-v) &= -E(v), &(12.7a)\\ F(-v) &= F(v). &(12.7b)\end{aligned}$$

(d) Reciprocity: The description of a process in the unprimed frame moving to the right with velocity v is identical to the description in the primed frame moving to the left.

Figure 12.3: Reciprocity.

That is, the transformation must be invariant under

$$(z,t) \to (z',t') \qquad (z',t') \to (z,t) \qquad v \to -v. \tag{12.8}$$

Show that this implies

$$\begin{aligned} z &= A(-v)\,(z' + vt'), &(12.9a)\\ t &= E(-v)\,z' + F(-v)\,t'. &(12.9b)\end{aligned}$$

Show that Eqs. (12.3) and Eqs. (12.9) imply

$$\begin{aligned} E(v) &= \frac{1}{v}\left[\frac{1}{A(v)} - A(v)\right], &(12.10a)\\ F(v) &= A(v). &(12.10b)\end{aligned}$$

(e) Together, for arbitrary $A(v)$, show that the relativity principle allows the following transformations,

$$z' = A(v)\,(z - vt), \tag{12.11a}$$
$$t' = A(v)\left[\frac{1}{v}\left(\frac{1}{A(v)^2} - 1\right) z + t\right]. \tag{12.11b}$$

i. In Galilean relativity we require $t' = t$. Show that this is obtained with

$$A(v) = 1 \tag{12.12}$$

in Eqs. (12.11). This leads to the Galilean transformation

$$z' = z - vt, \tag{12.13a}$$
$$t' = t. \tag{12.13b}$$

ii. In Einstein's special relativity the requirement is for a special speed c that is described identically by both the primed and unprimed frames. That is,

$$z = ct, \tag{12.14a}$$
$$z' = ct'. \tag{12.14b}$$

Show that Eqs. (12.14) when substituted in in Eqs. (12.11) leads to

$$A(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{12.15}$$

This corresponds to the Lorentz transformation

$$z' = A(v)(z - vt), \tag{12.16a}$$
$$t' = A(v)\left(-\frac{v}{c^2} z + t\right). \tag{12.16b}$$

iii. This suggests that it should be possible to contrive additional solutions for $A(v)$ that respects the relativity principle, but with new physical requirements for the respective choice of $A(v)$. Construct one such transformation. In particular, investigate modifications of Eqs. (12.14) that donot change the current experimental observations. The response to this part of the question will not be used for assessment.

12.2 Lorentz transformation

Problems

1. (**20 points.**) The Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}, \qquad \beta = \frac{v}{c}. \tag{12.17}$$

(a) Evaluate γ for $v = 30\,\text{m/s}$ ($\sim 70\,\text{miles/hour}$).

(b) Evaluate γ for $v = 3c/5$.

2. (**20 points.**) Lorentz transformation describing a boost in the x-direction is obtained using the matrix

$$L = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12.18}$$

(a) Show that the determinant of the matrix L is 1.

(b) Determine L^{-1}.

3. (**20 points.**) Lorentz transformation (in one dimension) is given by

$$\Delta z' = \gamma(\Delta z - v\Delta t), \tag{12.19a}$$
$$\Delta t' = \gamma\left(\Delta t - \frac{v}{c}\frac{\Delta z}{c}\right), \tag{12.19b}$$

where $\gamma = \sqrt{1 - v^2/c^2}$. Show that for

$$v \ll c \qquad \text{and} \qquad \frac{\Delta z}{\Delta t} \ll c \tag{12.20}$$

one obtains the Galilean transformation

$$\Delta z' = \Delta z - v\Delta t, \tag{12.21a}$$
$$\Delta t' = \Delta t. \tag{12.21b}$$

Note: For the case when Δz and Δt represent the change in position and time of a particle we could have v and $\Delta z/\Delta t$ to be identical.

4. (**20 points.**) How does the wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(z - ct) = 0 \tag{12.22}$$

transform under the Lorentz transformtion

$$z' = \gamma z + \beta\gamma ct, \tag{12.23a}$$
$$ct' = \beta\gamma z + \gamma ct. \tag{12.23b}$$

Solution:

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) f(a(z - ct)) = 0, \tag{12.24}$$

where $a = \sqrt{(1-\beta)/(1+\beta)}$.

5. (**20 points.**) Verify the following:

$$\mathrm{Tr}A = A_i{}^i. \tag{12.25a}$$
$$\det A = \varepsilon_{i_1 i_2 \dots i_n} A^{i_1}{}_1 A^{i_2}{}_2 \dots A^{i_n}{}_n \tag{12.25b}$$
$$= \frac{1}{n!}\varepsilon_{i_1 i_2 \dots i_n}\varepsilon^{i'_1 i'_2 \dots i'_n} A^{i_1}{}_{i'_1} A^{i_2}{}_{i'_2} \dots A^{i_n}{}_{i'_n}, \tag{12.25c}$$

where n is the dimension of the matrix A.

6. (**20 points.**) Prove that any orthogonal matrix R satisfying

$$RR^T = 1 \tag{12.26}$$

in N-dimensions has $N(N-1)/2$ independent variables.

7. (**20 points.**) Lorentz transformation describing a boost in the x-direction, y-direction, and z-direction, are

$$L_1 = \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 & 0 & 0 \\ -\beta_1\gamma_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} \gamma_2 & 0 & -\beta_2\gamma_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\beta_2\gamma_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_3 = \begin{pmatrix} \gamma_3 & 0 & 0 & -\beta_3\gamma_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_3\gamma_3 & 0 & 0 & \gamma_3 \end{pmatrix}, \tag{12.27}$$

respectively. Transformation describing a rotation about the x-axis, y-axis, and z-axis, are

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\omega_1 & \sin\omega_1 \\ 0 & 0 & -\sin\omega_1 & \cos\omega_1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_2 & 0 & -\sin\omega_2 \\ 0 & 0 & 1 & 0 \\ 0 & \sin\omega_2 & 0 & \cos\omega_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega_3 & \sin\omega_3 & 0 \\ 0 & -\sin\omega_3 & \cos\omega_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12.28}$$

respectively. For infinitesimal transformations, $\beta_i = \delta\beta_i$ and $\omega_i = \delta\omega_i$ use the approximations

$$\gamma_i \sim 1, \qquad \cos\omega_i \sim 1, \qquad \sin\omega_i \sim \delta\omega_i, \tag{12.29}$$

to identify the generator for boosts $\mathbf{N}$, and the generator for rotations the angular momentum $\mathbf{J}$,

$$\mathbf{L} = \mathbf{1} + \delta\boldsymbol{\beta}\cdot\mathbf{N} \qquad \text{and} \qquad \mathbf{R} = \mathbf{1} + \delta\boldsymbol{\omega}\cdot\mathbf{J}, \tag{12.30}$$

respectively. Then derive

$$\Big[N_1, N_2\Big] = N_1N_2 - N_2N_1 = J_3. \tag{12.31}$$

This states that boosts in perpendicular direction leads to rotation. (To gain insight of the statement, calculate $[J_1, J_2]$ and interpret the result.)

(a) Is velocity addition commutative?

(b) Is velocity addition associative?

(c) Read a resource article (Wikipedia) on Wigner rotation.

8. (**20 points.**) (Based on Hughston and Tod's book.) Prove the following.

(a) If p^μ is a time-like vector and $p^\mu s_\mu = 0$ then s^μ is necessarily space-like.

(b) If p^μ and q^μ are both time-like vectors and $p^\mu q_\mu < 0$ then either both are future-pointing or both are past-pointing.

(c) If p^μ and q^μ are both light-like vectors and $p^\mu q_\mu = 0$ then p^μ and q^μ are proportional.

(d) If p^μ is a light-like vector and $p^\mu s_\mu = 0$, then s^μ is space-like or p^μ and s^μ are proportional.

(e) If u^α, v^α, and w^α, are time-like vectors with $u^\alpha v_\alpha < 0$ and $v^\alpha w_\alpha < 0$, then $w^\alpha u_\alpha < 0$.

9. (**20 points.**) Non-relativistic limits are obtained for $\beta \ll 1$ in relativistic formulae.

(a) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$?

(b) Does Lorentz transformation recover Galilean transformation for $\beta \ll 1$ and $c \to \infty$?

12.3 Geometry of Lorentz transformation

1. (**20 points.**) A four-vector in the context of Lorentz tranformation can be described using the notation

$$a^\alpha = (a^0, a^1, a^2, a^3). \tag{12.32}$$

Let

$$b^\alpha = (b^0, b^1, b^2, b^3) \tag{12.33}$$

be another four-vector. The scalar product between two Lorentz vectors is given by

$$a^\alpha b_\alpha = -a^0 b^0 + a^1 b^1 + a^2 b^2 + a^3 b^3. \tag{12.34}$$

The square of the 'length' of the four-vector a^α is given by

$$a^\alpha a_\alpha, \tag{12.35}$$

which is not necessarily positive. The length of a four-vector is invariant, that is, it is independent of the Lorentz frame. If two Lorentz four-vectors are orthogonal they satisfy

$$a^\alpha b_\alpha = 0. \tag{12.36}$$

Orthogonality is an invariant concept.

(a) Determine the length of

$$p^\alpha = (5, 0, 0, 3), \tag{12.37}$$

where the numbers are in arbitrary units. Is it time-like, light-like, or space-like?

(b) Find a four-vector of the form

$$q^\alpha = (q^0, 0, 0, q^3) \tag{12.38}$$

that is perpendicular to p^α.

2. (**20 points.**) A hypothetical particle is observed by an inertial observer to be moving with non-uniform superluminal speed ($v > c$) at every instant of time from remote past to remote future. Draw a plausible world line of such a particle.

12.4 Poincaré (parallel) velocity addition formula

1. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is

$$v = \frac{v_a + v_b}{1 + \frac{v_a v_b}{c^2}}, \tag{12.39}$$

where v_a and v_b are velocities and c is speed of light in vacuum. Jerzy Kocik, from the department of Mathematics in SIUC, has invented a geometric diagram that allows one to visualize the Poincaré formula. (Refer [**2012Kocik**].) An interactive applet for exploring velocity addition is available at Kocik's web page [**2012Kocikwapp**]. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.)

(a) Analyse what is obtained if you add two subluminal speeds?

(b) Analyse what is obtained if you add a subluminal speed to speed of light?

(c) Analyse what is obtained if you add a subluminal speed to a superluminal speed?

(d) Analyse what is obtained if you add speed of light to another speed of light?

(e) Analyse what is obtained if you add a superluminal speed to speed of light?

(f) Analyse what is obtained if you add two superluminal speeds?

2. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,
$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{12.40}$$
where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the 'sum' subluminal or superluminal. Is the answer unique?

3. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is
$$v = \frac{v_a + v_b}{1 + \dfrac{v_a v_b}{c^2}}, \tag{12.41}$$
where v_a and v_b are velocities and c is speed of light in vacuum. (For the following assume that the Poincaré formula holds for all speeds, subluminal ($v_i < c$), superluminal ($v_i > c$), and speed of light.) Analyse what is obtained if you add a subluminal speed to a superluminal speed? That is, is the resultant speed subluminal or superluminal.
Hint: Analyse the case
$$\frac{v_a}{c} = -\frac{c}{v_b} \pm \delta, \tag{12.42}$$
for infinitely small $\delta > 0$.

4. (**20 points.**) The Poincaré formula for the addition of (parallel) velocities is, $c = 1$,
$$v = \frac{v_a + v_b}{1 + v_a v_b}, \tag{12.43}$$
where v_a and v_b are velocities and c is speed of light in vacuum. Assuming that the Poincaré formula holds for all speeds, subluminal ($-1 < v_i < 1$), superluminal ($|v_i| > 1$), and speed of light, analyse what is obtained if you add a speed to an infinitely large superluminal speed, that is, $v_b \to \infty$. Hint: Inversion.

5. (**30 points.**) Let
$$\tanh\theta = \beta, \tag{12.44}$$
where $\beta = v/c$. Addition of (parallel) velocities in terms of the parameter θ obeys the arithmatic addition
$$\theta = \theta_a + \theta_b. \tag{12.45}$$
   (a) Invert the expression in Eq. (12.44) to find the explicit form of θ in terms of β as a logarithm.
   (b) Show that Eq. (12.45) leads to the relation
$$\left(\frac{1+\beta}{1-\beta}\right) = \left(\frac{1+\beta_a}{1-\beta_a}\right)\left(\frac{1+\beta_b}{1-\beta_b}\right). \tag{12.46}$$
   (c) Using Eq. (12.46) derive the Poincaré formula for the addition of (parallel) velocities.

12.5 Kinematics

1. (**100 points.**) Relativisitic kinematics is constructed in terms of the proper time element ds, which remains unchanged under a Lorentz transformation,
$$-\,ds^2 = -c^2 dt^2 + d\mathbf{x}\cdot d\mathbf{x}. \tag{12.47}$$
Here $\mathbf{x}$ and t are the position and time of a particle. They are components of a vector under Lorentz transformation and together constitute the position four-vector
$$x^\alpha = (ct, \mathbf{x}). \tag{12.48}$$

(a) Velocity: The four-vector associated with velocity is constructed as

$$u^\alpha = c\frac{dx^\alpha}{ds}. \tag{12.49}$$

i. Using Eq. (12.47) deduce

$$\gamma ds = cdt, \quad \text{where} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \frac{d\mathbf{x}}{dt}. \tag{12.50}$$

Then, show that

$$u^\alpha = (c\gamma, \mathbf{v}\gamma). \tag{12.51}$$

Here $\mathbf{v}$ is the velocity that we use in Newtonian physics.

ii. Further, show that

$$u^\alpha u_\alpha = -c^2. \tag{12.52}$$

Thus, conclude that the velocity four-vector is a time-like vector. What is the physical implication of this statement for a particle?

iii. Write down the form of the velocity four-vector in the rest frame of the particle?

(b) Momentum: Define momentum four-vector in terms of the mass m of the particle as

$$p^\alpha = mu^\alpha = (mc\gamma, m\mathbf{v}\gamma). \tag{12.53}$$

Connection with the physical quantities associated to a moving particle, the energy and momentum of the particle, is made by identifying (or defining)

$$p^\alpha = \left(\frac{E}{c}, \mathbf{p}\right), \tag{12.54}$$

which corresponds to the definitions

$$E = mc^2\gamma, \tag{12.55a}$$
$$\mathbf{p} = m\mathbf{v}\gamma, \tag{12.55b}$$

for energy and momentum, respectively. Discuss the non-relativistic limits of these quantities. In particular, using the approximation

$$\gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots, \tag{12.56}$$

show that

$$E - mc^2 = \frac{1}{2}mv^2 + \ldots, \tag{12.57a}$$
$$\mathbf{p} = m\mathbf{v} + \ldots. \tag{12.57b}$$

Evaluate

$$p^\alpha p_\alpha = -m^2c^2. \tag{12.58}$$

Thus, derive the energy-momentum relation

$$E^2 - p^2c^2 = m^2c^4. \tag{12.59}$$

(c) Acceleration: The four-vector associated with acceleration is constructed as

$$a^\alpha = c\frac{du^\alpha}{ds}. \tag{12.60}$$

i. Show that
$$a^\alpha = \gamma \left(c\frac{d\gamma}{dt}, \mathbf{v}\frac{d\gamma}{dt} + \gamma\mathbf{a} \right), \tag{12.61}$$
where
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} \tag{12.62}$$
is the acceleration that we use in Newtonian physics.

ii. Starting from Eq. (12.52) and taking derivative with respect to proper time show that
$$u^\alpha a_\alpha = 0. \tag{12.63}$$
Thus, conclude that four-acceleration is space-like.

iii. Further, using the explicit form of $u^\alpha a_\alpha$ in Eq. (12.63) derive the identity
$$\frac{d\gamma}{dt} = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\right)\gamma^3. \tag{12.64}$$

iv. Show that
$$a^\alpha = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4, \mathbf{a}\gamma^2 + \frac{\mathbf{v}}{c}\frac{\mathbf{v}\cdot\mathbf{a}}{c}\gamma^4 \right) \tag{12.65}$$

v. Write down the form of the acceleration four-vector in the rest frame ($\mathbf{v} = 0$) of the particle as $(0, \mathbf{a}_0)$, where
$$\mathbf{a}_0 = \mathbf{a}\big|_{\text{rest frame}} \tag{12.66}$$
is defined as the proper acceleration. Note that the proper acceleration is a Lorentz invariant quantity, that is, independent of which observer makes the measurement.

vi. Evaluate the following identities involving the proper acceleration
$$a^\alpha a_\alpha = \mathbf{a}_0\cdot\mathbf{a}_0 = \left[\mathbf{a}\cdot\mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c}\right)^2\gamma^2\right]\gamma^4 = \left[\mathbf{a}\cdot\mathbf{a} - \left(\frac{\mathbf{v}\times\mathbf{a}}{c}\right)^2\right]\gamma^6. \tag{12.67}$$

vii. In a particular frame, if $\mathbf{v}\,||\,\mathbf{a}$ (corresponding to linear motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^3. \tag{12.68}$$
And, in a particular frame, if $\mathbf{v}\perp\mathbf{a}$ (corresponding to circular motion), deduce
$$|\mathbf{a}_0| = |\mathbf{a}|\gamma^2. \tag{12.69}$$

(d) Force: The force four-vector is defined as
$$f^\alpha = c\frac{dp^\alpha}{ds} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \mathbf{F}\gamma\right), \tag{12.70}$$
where the force $\mathbf{F}$, identified (or defined) as
$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \tag{12.71}$$
is the force in Newtonian physics. Starting from Eq. (12.58) derive the relation
$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} \tag{12.72}$$
which is the power output or the rate of work done by the force $\mathbf{F}$ on the particle.

(e) Equations of motion: The relativistic generalization of Newton's laws are

$$f^\alpha = m a^\alpha. \tag{12.73}$$

Show that these involve the relations, using the definitions of energy and momentum in Eqs. (12.55),

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}\gamma + m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\gamma^3, \tag{12.74a}$$

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v} = m\mathbf{v}\cdot\mathbf{a}\gamma^3. \tag{12.74b}$$

Discuss the non-relativistic limits of the equations of motion.

2. (**20 points.**) Lorentz transformation relates the energy E and momentum $\mathbf{p}$ of a particle when measured in different frames. For example, for the special case when the relative velocity and the velocity of the particle are parallel we have

$$\begin{pmatrix} E'/c \\ p' \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E/c \\ p \end{pmatrix}. \tag{12.75}$$

Photons are massless spin 1 particles whose energy and momentum are $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, such that $\omega = kc$. Thus, derive the relativistic Doppler effect formula

$$\omega' = \omega\sqrt{\frac{1+\beta}{1-\beta}}. \tag{12.76}$$

Contrast the above formula with the Doppler effect formula for sound.

3. (**20 points.**) Neutral π meson decays into two photons. That is,

$$\pi^0 \to \gamma_1 + \gamma_2. \tag{12.77}$$

Energy-momentum conservation for the decay in the laboratory frame, in which the meson is not necessarily at rest, is given by

$$p_\pi^\alpha = p_1^\alpha + p_2^\alpha. \tag{12.78}$$

Or, more specifically,

$$\left(\frac{E_\pi}{c}, \mathbf{p}\right) = \left(\frac{E_1}{c}, \mathbf{p}_1\right) + \left(\frac{E_2}{c}, \mathbf{p}_2\right), \tag{12.79}$$

where E_π and $\mathbf{p}$ are the energy and momentum of neutral π meson, and E_i's and $\mathbf{p}_i$'s are the energies and momentums of the photons. Thus, derive the relation

$$m_\pi^2 c^4 = 2E_1E_2(1-\cos\theta), \tag{12.80}$$

where m_π is the mass of neutral π meson, and θ is the angle between the directions of $\mathbf{p}_1$ and $\mathbf{p}_2$.

4. (**20 points.**) Using Maxwell's equations we can show that a monochromatic electromagnetic wave has the electromagnetic energy density U and electromagnetic momentum density $\mathbf{G}$ given by

$$U = \frac{1}{2}\varepsilon_0^2E^2 + \frac{1}{2}\mu_0^2H^2 = \varepsilon_0^2E^2 = \mu_0^2H^2, \tag{12.81}$$

$$\mathbf{G} = \frac{\mathbf{E}\times\mathbf{H}}{c^2} = \hat{\mathbf{k}}\frac{U}{c}. \tag{12.82}$$

Observe that are densities. The energy and momentum densities do not transform like a four-vector, instead they are part of a four-tensor,

$$t^{\alpha\beta} = \begin{pmatrix} cU, & \mathbf{S} \\ c^2\mathbf{G}, & c\mathbf{T} \end{pmatrix}. \tag{12.83}$$

Note: Complete this!

5. (**20 points.**) Length contracts and time dilates. That is,

$$L = \frac{L_0}{\gamma}, \qquad T = T_0\gamma, \tag{12.84}$$

where L_0 and T_0 are proper length and proper time. Similarly, show that (for $\mathbf{v}\,||\,\mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{12.85}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneaous rest frame of the particle. Further, for $\mathbf{v} \perp \mathbf{a}$ show that

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^2}. \tag{12.86}$$

6. (**20 points.**) Time dilates. That is,

$$T = T_0\gamma, \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}, \tag{12.87}$$

where T_0 is the proper time measured in the instantaneous rest frame of the clock measuring T_0 and T is the time measured by a clock moving with velocity v relative to the clock measuring proper time. Similarly, show that (for $\mathbf{v}\,||\,\mathbf{a}$)

$$|\mathbf{a}| = \frac{|\mathbf{a}_0|}{\gamma^3}, \tag{12.88}$$

where $|\mathbf{a}_0|$ is the proper acceleration measured in the instantaneous rest frame of the particle. Derive the equation for the trajectory of a particle moving in a straight line (along the z axis) with constant proper acceleration, after starting from rest from the point $z = c^2/|\mathbf{a}_0|$ at time $t = 0$.

12.6 Dynamics

12.6.1 Charge particle in a uniform magnetic field: Circular motion

1. (**20 points.**) A relativisitic particle in a uniform magnetic field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{12.89a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{12.89b}$$

where

$$E = mc^2\gamma, \tag{12.90a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{12.90b}$$

and

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B}. \tag{12.91}$$

Show that

$$\frac{d\gamma}{dt} = 0. \tag{12.92}$$

Then, derive

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}\times\boldsymbol{\omega}_c, \tag{12.93}$$

where

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}}{m\gamma}. \tag{12.94}$$

Compare this relativistic motion to the associated non-relativistic motion.

2. (**20 points.**) If the motion of a non-relativistic particle is such that it does not change the kinetic energy of the particle, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = 0. \tag{12.95}$$

Show that this imples

$$\mathbf{v}\cdot\mathbf{a} = 0. \tag{12.96}$$

This is achieved when the acceleration $a = 0$ or in the case of uniform circular motion. Starting from Eq. (12.96) show that the relativistic generalization of kinetic energy $E = mc^2\gamma$ is also conserved, that is,

$$\frac{d}{dt}(mc^2\gamma) = 0. \tag{12.97}$$

Observe that

$$\boldsymbol{\beta}\cdot\mathbf{a} = \frac{d}{dt}\left(\frac{\beta^2}{2}\right) = -\frac{1}{2}\frac{d}{dt}\frac{1}{\gamma^2} = \frac{1}{\gamma^3}\frac{d\gamma}{dt}. \tag{12.98}$$

12.6.2 Charge particle in a uniform electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F}\cdot\mathbf{v}, \tag{12.99a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{12.99b}$$

where

$$E = mc^2\gamma, \tag{12.100a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{12.100b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{12.101}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{12.102}$$

and initial conditions

$$\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{12.103a}$$

$$\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}. \tag{12.103b}$$

(a) In terms of the definition

$$\boldsymbol{\omega}_0 = \frac{1}{c}\frac{q\mathbf{E}}{m}, \tag{12.104}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0\cdot\boldsymbol{\beta} \tag{12.105}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{12.106}$$

(b) Since the particle starts from rest show that we have

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t. \tag{12.107}$$

For our configuration this implies

$$\beta_x = 0, \tag{12.108a}$$
$$\beta_y \gamma = \omega_0 t, \tag{12.108b}$$
$$\beta_z = 0. \tag{12.108c}$$

Further, deduce

$$\beta_y = \frac{\omega_0 t}{\sqrt{1+\omega_0^2 t^2}}. \tag{12.109}$$

Integrate again and use the initial condition to show that the motion is described by

$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right]. \tag{12.110}$$

Rewrite the solution in the form

$$\left(y - y_0 + \frac{c}{\omega_0}\right)^2 - c^2 t^2 = \frac{c^2}{\omega_0^2}. \tag{12.111}$$

This represents a hyperbola passing through $y = y_0$ at $t = 0$. If we choose the initial position $y_0 = c/\omega_0$ we have

$$y^2 - c^2 t^2 = y_0^2. \tag{12.112}$$

(c) The (constant) proper acceleration associated with this motion is

$$\alpha = \omega_0 c = \frac{c^2}{y_0}. \tag{12.113}$$

A Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$y - y_0 = \frac{1}{2}\alpha t^2. \tag{12.114}$$

Show that the hyperbolic curve

$$y = y_0\sqrt{1+\frac{c^2 t^2}{y_0^2}} \tag{12.115}$$

in regions that satisfy

$$\omega_0 t \ll 1 \tag{12.116}$$

is approximately the parabolic curve

$$y = y_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{12.117}$$

2. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by equation of a hyperbola

$$z^2 - c^2 t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{12.118}$$

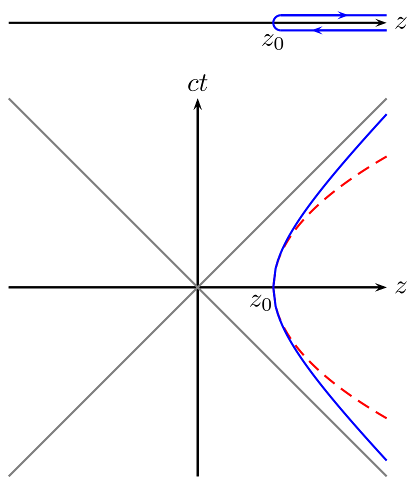

Figure 12.4: Problem 2

(a) This represents the world-line of a particle thrown from $z > z_0$ at $t < 0$ towards $z = z_0$ in region of constant (proper) acceleration α as described by the bold (blue) curve in the space-time diagram in Figure 2. In contrast a Newtonian particle moving with constant acceleration α is described by equation of a parabola

$$z - z_0 = \frac{1}{2}\alpha t^2 \tag{12.119}$$

as described by the dashed (red) curve in the space-time diagram in Figure 2. Show that the hyperbolic curve

$$z = z_0\sqrt{1 + \frac{c^2 t^2}{z_0^2}} \tag{12.120}$$

in regions that satisfy

$$t \ll \frac{c}{\alpha} \tag{12.121}$$

is approximately the parabolic curve

$$z = z_0 + \frac{1}{2}\alpha t^2 + \ldots. \tag{12.122}$$

(b) Recognize that the proper acceleration α does not have an upper bound.

(c) A large acceleration is achieved by taking an above turn while moving very fast. Thus, turning around while moving close to the speed of light c should achieve the highest acceleration. Show that $\alpha \to \infty$ corresponding to $z_0 \to 0$ represents this scenario. What is the equation of motion of a particle moving with infinite proper acceleration. To gain insight, plot world-lines of particles moving with $\alpha = c^2/z_0$, $\alpha = 10c^2/z_0$, and $\alpha = 100c^2/z_0$.

3. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration α is described by the equation of a hyperbola

$$z^2 - c^2 t^2 = z_0^2, \qquad z_0 = \frac{c^2}{\alpha}. \tag{12.123}$$

This is the motion of a particle 'dropped' from $z = z_0$ at $t = 0$ in region of constant (proper) acceleration. See Figure 3. Using geometric (diagrammatic) arguments might be easiest to answer the following.

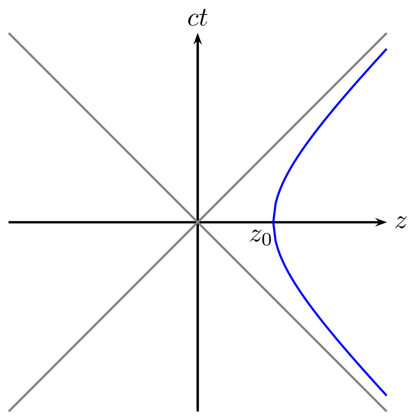

Figure 12.5: Problem 3

(a) Will a photon dispatched to 'chase' this particle at $t = 0$ from $z = 0$ ever catch up with it? If yes, when and where does it catch up?

(b) Will a photon dispatched to 'chase' this particle at $t = 0$ from $0 < z < z_0$ ever catch up with it? If yes, when and where does it catch up?

(c) Will a photon dispatched to 'chase' this particle, at $t = 0$ from $z < 0$ ever catch up with it? If yes, when and where does it catch up?

What are the implications for the observable part of our universe from this analysis?

4. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_q(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{12.124}$$

This is the motion of a particle that comes to existance at $z_q = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Assume you are positioned at the origin. If the particle is a source of light (imagine a flash light) at what time will the light first reach you at the origin? Where is the particle when this happens?

5. (**20 points.**) The path of a relativistic particle moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_2(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{12.125}$$

This is the motion of a particle that comes to existance at $z_2 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_2(0) = 0$ and $y_2(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Another particle is at rest at z_1

$$z_1(t) = z_1, \tag{12.126}$$

such that $0 < z_1 < z_0$. Assume that both particles emit photons continuously.

(a) At what time do photons emitted by 2 first reach 1? Where is particle 2 when this happens?

(b) At what time is the last photon that reaches 2 emitted by 1? Where is particle 2 when this happens?

(c) Do all the photons emitted by 1 reach 2?

(d) Do all the photons emitted by 2 reach 1?

6. (**20 points.**) The path of a relativistic particle 1 moving along a straight line with constant (proper) acceleration g is described by the equation of a hyperbola

$$z_1(t) = \sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{12.127}$$

This is the motion of a particle that comes to existance at $z_1 = +\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = z_0$, and then returns back to infinity. Consider another relavistic particle 2 undergoing hyperbolic motion given by

$$z_2(t) = -\sqrt{c^2t^2 + z_0^2}, \qquad z_0 = \frac{c^2}{g}. \tag{12.128}$$

This is the motion of a particle that comes to existance at $z_2 = -\infty$ at $t = -\infty$, then 'falls' with constant (proper) acceleration g. If we choose $x_q(0) = 0$ and $y_q(0) = 0$, the particle 'falls' keeping itself on the z-axis, comes to stop at $z = -z_0$, and then returns back to negative infinity. The world-line of particle 1 is the blue curve in Figure 6, and the world-line of particle 2 is the red curve in Figure 6. Using geometric (diagrammatic) arguments might be easiest to answer the following. Imagine the particles are sources of light (imagine a flash light pointing towards origin).

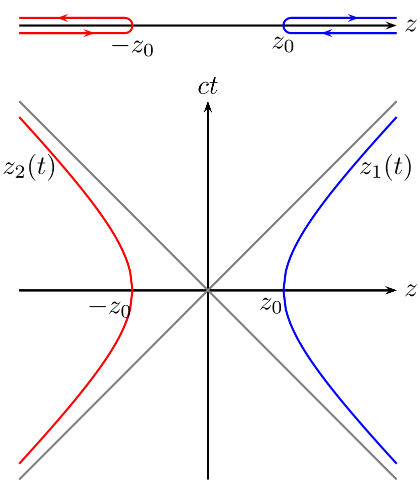

Figure 12.6: Problem 6

(a) At what time will the light from particle 1 first reach particle 2? Where are the particles when this happens?

(b) At what time will the light from particle 2 first reach particle 1? Where are the particles when this happens?

(c) Can the particles communicate with each other?

(d) Can the particles ever detect the presence of the other? In other words, can one particle be aware of the existence of the other? What can you deduce about the observable part of our universe from this analysis?

7. (**20 points.**) Two masses (one heavier than the other) move with constant proper acceleration α, after they are dropped from position $x_0 = c^2/\alpha$. Does the time taken to fall a given distance depend on mass?

Recall that Aristotle (384-322 BC) presumed that the time taken to fall a given distance depended on mass. Galileo (1564-1642) argued, based on a famous thought experiment (refer Wikipedia) that the time taken to fall a given distance is independent of mass.

(a) Consider an electron and a proton connected by a hypothetical string. What is the tension in the string when they move in a uniform electric field (which leads to proper acceleration). We will have to dictate how the distance between them changes.

(b) What about charges of different masses in an electric field?

(c) What about a hydrogen atom? How does electrostatic energy associated to the hydrogen atom fall?

(d) Do these considerations involve a Poincare stress?

Keywords: Trouton-Noble experiment, Laue current, 4/3 problem.
NOTE: This problem needs thought and scrutiny!

12.6.3 Charge particle in a uniform electric field with an initial velocity normal to electric field: Hyperbolic motion

1. (**20 points.**) A relativisitic particle in a uniform electric field is described by the equations

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{12.129a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{12.129b}$$

where

$$E = mc^2\gamma, \tag{12.130a}$$

$$\mathbf{p} = m\mathbf{v}\gamma, \tag{12.130b}$$

and

$$\mathbf{F} = q\mathbf{E}. \tag{12.131}$$

Let us consider the configuration with the electric field in the $\hat{\mathbf{y}}$ direction,

$$\mathbf{E} = E\,\hat{\mathbf{y}}, \tag{12.132}$$

and initial conditions

$$\mathbf{v}(0) = v_0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}, \tag{12.133a}$$

$$\mathbf{x}(0) = x_0\,\hat{\mathbf{x}} + y_0\,\hat{\mathbf{y}} + z_0\,\hat{\mathbf{z}}. \tag{12.133b}$$

We will use the associated definitions $\boldsymbol{\beta}_0 = \mathbf{v}(0)/c$ and $\boldsymbol{\gamma}_0 = 1/\sqrt{1-\beta_0^2}$.

(a) In terms of the definition

$$\boldsymbol{\alpha} = \boldsymbol{\omega}_0 c = \frac{q\mathbf{E}}{m}, \tag{12.134}$$

show that the equations of motion are given by

$$\frac{d\gamma}{dt} = \boldsymbol{\omega}_0 \cdot \boldsymbol{\beta} \tag{12.135}$$

and

$$\frac{d}{dt}(\boldsymbol{\beta}\gamma) = \boldsymbol{\omega}_0. \tag{12.136}$$

(b) For our configuration show that

$$\boldsymbol{\beta}\gamma = \boldsymbol{\omega}_0 t + \beta_0\gamma_0\hat{\mathbf{x}}, \tag{12.137}$$

such that

$$\beta_x\gamma = \beta_0\gamma_0, \tag{12.138a}$$
$$\beta_y\gamma = \omega_0 t, \tag{12.138b}$$
$$\beta_z\gamma = 0. \tag{12.138c}$$

Using $\beta_z\gamma = 0$, learn that

$$\frac{\beta_z^2}{1-\beta_x^2-\beta_y^2-\beta_z^2} = 0 \tag{12.139}$$

and in conjunction with $\beta_x\gamma = \beta_0\gamma_0$ deduce that

$$\beta_z = 0 \tag{12.140}$$

and

$$\frac{\beta_x^2}{\beta_0^2} + \beta_y^2 = 1. \tag{12.141}$$

Thus, deduce

$$\gamma^2 = \omega_0^2 t^2 + \gamma_0^2 \tag{12.142}$$

and

$$\beta_x^2 + \beta_y^2 = \beta_0^2 + \frac{\beta_y^2}{\gamma_0^2}. \tag{12.143}$$

Further, deduce

$$\beta_y = \frac{\bar{\omega}_0 t}{\sqrt{1+\bar{\omega}_0^2 t^2}} \tag{12.144}$$

and

$$\beta_x = \frac{\beta_0}{\sqrt{1+\bar{\omega}_0^2 t^2}}, \tag{12.145}$$

where

$$\bar{\omega}_0 = \frac{\omega_0}{\gamma_0}. \tag{12.146}$$

Integrate again and use the initial condition to show that the motion is described by

$$x - x_0 = \frac{v_0}{\bar{\omega}_0}\sinh^{-1}\bar{\omega}_0 t, \tag{12.147a}$$
$$y - y_0 = \frac{c}{\bar{\omega}_0}\left[\sqrt{1+\bar{\omega}_0^2 t^2} - 1\right], \tag{12.147b}$$
$$z - z_0 = 0. \tag{12.147c}$$

(c) Show that for $v_0 = 0$ we reproduce the solution for a particle starting from rest. Next, for

$$\bar{\omega}_0 t \ll 1 \tag{12.148}$$

and

$$\alpha = \bar{\omega}_0 c \tag{12.149}$$

obtain the non-relativistic limits,

$$x - x_0 = v_0 t, \tag{12.150a}$$
$$y - y_0 = \frac{1}{2}\alpha t^2, \tag{12.150b}$$
$$z - z_0 = 0. \tag{12.150c}$$

Hint: Recall the series expansion

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right) = x + \dots. \tag{12.151}$$

(d) For the choice of initial position,

$$x_0 = 0, \qquad y_0 = \frac{c}{\bar{\omega}_0} = \frac{c^2\gamma_0}{\alpha}, \qquad z_0 = 0, \tag{12.152}$$

show that the trajectory is a catenary,

$$y = y_0 \cosh\left(\frac{\bar{\omega}_0}{v_0} x\right). \tag{12.153}$$

Chapter 13

Lorentz covariance of electrodynamic quantities

13.1 Maxwell equations

1. In terms of the four-vector potential
$$cA^\mu = (\phi, c\mathbf{A}) \tag{13.1}$$
the Maxwell field tensor $F_{\mu\nu}$ is defined as
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{13.2}$$
Note that, by construction, the field tensor is antisymmetric. Recall,
$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right). \tag{13.3}$$
Using the expression for the electric and magnetic field in terms of the potentials,
$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{13.4a}$$
$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{13.4b}$$
in Eq. (13.2), recognize
$$cF_{0i} = -E_i \tag{13.5}$$
and
$$F_{ij} = \varepsilon_{ijk}B^k. \tag{13.6}$$
The tensor structure is more explicitly visualized in the form
$$cF_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}. \tag{13.7}$$

2. In terms of the four-current
$$j^\mu = (c\rho, \mathbf{j}), \tag{13.8}$$
show that the inhomogeneous Maxwell equations,
$$\boldsymbol{\nabla}\cdot\varepsilon_0\mathbf{E} = \rho, \tag{13.9a}$$
$$\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial}{\partial t}\varepsilon_0\mathbf{E} = \mathbf{j}, \tag{13.9b}$$

where $\mathbf{B} = \mu_0 \mathbf{H}$, are summarized in the covariant equation

$$\partial_\beta F^{\alpha\beta} = \mu_0 j^\alpha. \tag{13.10}$$

3. Show that

$$\partial_\alpha \partial_\beta F^{\alpha\beta} = 0, \tag{13.11}$$

using the antisymmetry of the field tensor. Thus, derive

$$\partial_\alpha j^\alpha = 0, \tag{13.12}$$

and recognize it as the statement of conservation of charge,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0, \tag{13.13}$$

in covariant form.

4. The dual Maxwell field tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{13.14}$$

where the total antisymmetrical tensor of the fourth rank is normalized to

$$\varepsilon^{0123} = +1. \tag{13.15}$$

Show that

$$\tilde{F}_{0i} = -B_i \tag{13.16}$$

and

$$\tilde{F}_{ij} = -\varepsilon_{ijk} E^k. \tag{13.17}$$

The dual field tensor is more explicitly visualized in the form

$$c\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & -cB_1 & -cB_2 & -cB_3 \\ cB_1 & 0 & -E_3 & E_2 \\ cB_2 & E_3 & 0 & -E_1 \\ cB_3 & -E_2 & E_1 & 0 \end{pmatrix}. \tag{13.18}$$

Using antisymmetry derive

$$\partial_\beta \tilde{F}^{\alpha\beta} = 0 \tag{13.19}$$

and show that it summarizes the homogeneous Maxwell equations,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{13.20a}$$

$$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{13.20b}$$

in covariant form.

5. Show that

$$-j^\alpha A_\alpha = \rho\phi - \mathbf{j} \cdot \mathbf{A}, \tag{13.21}$$

and recognize this as the electrodynamic interaction energy in covariant form.

6. In terms of the four-velocity

$$u^\alpha = (c\gamma, \mathbf{v}\gamma) \tag{13.22}$$

and four-momentum

$$p^\alpha = (\frac{E}{c}, \mathbf{p}) = m u^\alpha \tag{13.23}$$

show that the covariant Lorentz force equation is

$$\frac{dp^\alpha}{ds} = qF^{\alpha\beta} u_\beta. \tag{13.24}$$

In particular, show that

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v}, \tag{13.25a}$$

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}, \tag{13.25b}$$

where

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}. \tag{13.26}$$

13.2 Conservation equations

Show that

$$F^{\mu\nu} j_\nu + \partial_\nu t^{\mu\nu} = 0. \tag{13.27}$$

Identify the energy-momentum stress tensor

$$t^{\mu\nu} = F^{\mu\lambda} F^\nu{}_\lambda + g^{\mu\nu} \mathcal{L}, \tag{13.28}$$

where

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{13.29}$$

13.3 Lorentz invariant constructions

1. (**20 points.**) In terms of the four-vector potential

$$cA^\mu = (\phi, c\mathbf{A}) \tag{13.30}$$

the Maxwell field tensor $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{13.31}$$

and the corresponding dual tensor is defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}. \tag{13.32}$$

Derive the following relations, which involve quantities that remain invariant under Lorentz transformations.

$$c^2 F^{\mu\nu} F_{\mu\nu} = 2(c^2 B^2 - E^2). \tag{13.33a}$$

$$c^2 \tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} = -2(c^2 B^2 - E^2). \tag{13.33b}$$

$$c^2 F^{\mu\nu} \tilde{F}_{\mu\nu} = 4\, c\mathbf{B} \cdot \mathbf{E}. \tag{13.33c}$$

2. (**20 points.**) Eigenvalues of the energy momentum tensor. (If we choose $c = 1$, which is easily undone by replacing $\mathbf{E} \to \frac{1}{c}\mathbf{E}$ everywhere.)

   (a) Using

$$cF_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & cB_3 & -cB_2 \\ E_2 & -cB_3 & 0 & cB_1 \\ E_3 & cB_2 & -cB_1 & 0 \end{pmatrix}. \tag{13.34}$$

   and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{13.35}$$

   derive

$$cF^{\mu\lambda}c\tilde{F}_{\lambda\nu} = \delta^\mu{}_\nu \mathbf{E}\cdot c\mathbf{B}, \tag{13.36a}$$

$$c\tilde{F}^{\mu\lambda}c\tilde{F}_{\lambda\nu} - cF^{\mu\lambda}cF_{\lambda\nu} = \delta^\mu{}_\nu(c^2B^2 - E^2). \tag{13.36b}$$

   (b) Define

$$\mathcal{L} = \frac{\varepsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad \text{and} \quad \mathcal{G} = \varepsilon_0\mathbf{E}\cdot c\mathbf{B}, \tag{13.37}$$

   such that

$$-2\mu_0c^2\mathcal{L} = c^2B^2 - E^2 \quad \text{and} \quad \mu_0c^2\mathcal{G} = \mathbf{E}\cdot c\mathbf{B}. \tag{13.38}$$

   Thus, construct matrix (or dyadic) equations

$$\mathbf{F}\cdot\tilde{\mathbf{F}} = \mu_0\mathcal{G}\mathbf{1}, \tag{13.39a}$$

$$\tilde{\mathbf{F}}\cdot\tilde{\mathbf{F}} - \mathbf{F}\cdot\mathbf{F} = -2\mu_0\mathcal{L}\mathbf{1}, \tag{13.39b}$$

   in terms of matrices (or dyadics) $\mathbf{F}$ and $\tilde{\mathbf{F}}$.

   (c) Show that the eigenvalues λ of the field tensor $\mathbf{F}/\sqrt{\mu_0}$ satisfy the quartic equation

$$\lambda^4 - 2\mathcal{L}\lambda^2 - \mathcal{G}^2 = 0. \tag{13.40}$$

   (d) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{\mathcal{L} - \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}, \tag{13.41a}$$

$$\lambda_2 = \sqrt{\mathcal{L} + \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}. \tag{13.41b}$$

3. (**20 points.**) The eigenvalues λ of the field tensor $F^{\mu\nu}/\sqrt{\mu_0}$ satisfy the quartic equation

$$\lambda^4 - 2\mathcal{L}\lambda^2 - \mathcal{G}^2 = 0 \tag{13.42}$$

   in terms of

$$\mathcal{L} = \frac{\varepsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} \quad \text{and} \quad \mathcal{G} = \varepsilon_0\mathbf{E}\cdot c\mathbf{B}, \tag{13.43}$$

   such that

$$-2\mu_0c^2\mathcal{L} = c^2B^2 - E^2 \quad \text{and} \quad \mu_0c^2\mathcal{G} = \mathbf{E}\cdot c\mathbf{B}. \tag{13.44}$$

   (a) Evaluate the eigenvalues to be $\pm\lambda_1$ and $\pm\lambda_2$ where

$$\lambda_1 = \sqrt{\mathcal{L} - \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}, \tag{13.45a}$$

$$\lambda_2 = \sqrt{\mathcal{L} + \sqrt{\mathcal{L}^2 + \mathcal{G}^2}}. \tag{13.45b}$$

(b) In terms of the complex field

$$c\mathbf{X} = \frac{\mathbf{E} + ic\mathbf{B}}{\sqrt{2}} \tag{13.46}$$

show that

$$\mathcal{Z} = \frac{1}{\mu_0}\mathbf{X}\cdot\mathbf{X} = \mathcal{L} + i\mathcal{G} \tag{13.47}$$

and

$$\mathcal{Z}^* = \mathcal{L} - i\mathcal{G}. \tag{13.48}$$

Then, express the eigenvalues as

$$\frac{\lambda}{\sqrt{\mu_0}} = \pm\frac{1}{\sqrt{2}}\left(\sqrt{\mathcal{L}+i\mathcal{G}} \pm \sqrt{\mathcal{L}-i\mathcal{G}}\right). \tag{13.49}$$

Hint: Substitute $\mathcal{Z} = Re^{i\theta}$.

(c) Show that

i. if $c^2B^2 - E^2 = 0$, then the eigenvalues are $\pm\sqrt{\mathcal{G}}$ and $\pm i\sqrt{\mathcal{G}}$.

ii. if $\mathbf{E}\cdot c\mathbf{B} = 0$, then the eigenvalues are 0, 0, and $\pm\sqrt{2\mathcal{L}}$.

(d) Is the following true?

i. There is no Lorentz transformation connecting two reference frames such that the field is purely magnetic in origin in one and purely electric in origin in the other.

ii. If $c^2B^2 - E^2 > 0$ in a frame, then there exists a frame in which the field is purely magnetic.

iii. If $c^2B^2 - E^2 < 0$ in a frame, then there exists a frame in which the field is purely electric.

iv. If $c^2B^2 - E^2 = 0$ in a frame, then it is so in every frame.

v. $\mathbf{E}\cdot c\mathbf{B} > 0$ in a frame, then there exists a frame in which the fields are parallel.

vi. $\mathbf{E}\cdot c\mathbf{B} < 0$ in a frame, then there exists a frame in which the fields are antiparallel.

vii. $\mathbf{E}\cdot c\mathbf{B} = 0$ in a frame, then it is so in every frame.

viii. An electromagnetic plane wave is characterized by $c^2B^2 - E^2 = 0$ and $\mathbf{E}\cdot c\mathbf{B} = 0$.

4. (**40 points.**) The electric and magnetic fields transform under a Lorentz transformation (for boost in z direction) as

$$E'_x(\mathbf{r}',t') = \gamma\, E_x(\mathbf{r},t) + \beta\gamma\, cB_y(\mathbf{r},t), \tag{13.50a}$$
$$cB'_y(\mathbf{r}',t') = \beta\gamma\, E_x(\mathbf{r},t) + \gamma\, cB_y(\mathbf{r},t), \tag{13.50b}$$
$$E'_z(\mathbf{r}',t') = E_z(\mathbf{r},t) \tag{13.50c}$$

$$cB'_x(\mathbf{r}',t') = \gamma\, cB_x(\mathbf{r},t) - \beta\gamma\, E_y(\mathbf{r},t), \tag{13.51a}$$
$$E'_y(\mathbf{r}',t') = -\beta\gamma\, cB_x(\mathbf{r},t) + \gamma\, E_y(\mathbf{r},t), \tag{13.51b}$$
$$cB'_z(\mathbf{r}',t') = cB_z(\mathbf{r},t), \tag{13.51c}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. The transformed values of the coordinates and the fields are distinguished by a prime. Derive the invariance properties

$$\mathbf{E}'(\mathbf{r}',t')\cdot\mathbf{B}'(\mathbf{r}',t') = \mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t) \tag{13.52}$$

and

$$\mathbf{E}'(\mathbf{r}',t')^2 - c^2\mathbf{B}'(\mathbf{r}',t')^2 = \mathbf{E}(\mathbf{r},t)^2 - c^2\mathbf{B}(\mathbf{r},t)^2. \tag{13.53}$$

5. (**20 points.**) Let an infinitely thin plate occupying the $y = 0$ plane consist of a uniform charge density flowing in the $\hat{\mathbf{x}}$ direction described by drift velocity $\beta_d = v/c$.

(a) Show that the electric and magnetic field for this configuration is given by

$$\mathbf{E} = \eta(y)\,\hat{\mathbf{y}}\,\frac{\sigma}{2\varepsilon_0}, \tag{13.54a}$$
$$c\mathbf{B} = \eta(y)\,\hat{\mathbf{z}}\,\beta_d E, \tag{13.54b}$$

where

$$\eta(y) = \begin{cases} 1, & y > 0, \\ -1, & y < 0. \end{cases} \tag{13.55}$$

Thus, we have

$$cB = \beta_d E. \tag{13.56}$$

Recall that the motion of a point charge in this field configuration is a cycloid,

$$x(t) - v_q t = R \sin \omega_c t, \tag{13.57a}$$

$$y(t) - R = R \cos \omega_c t, \tag{13.57b}$$

that satisfies

$$\big[x(t) - v_q t\big]^2 + \big[y(t) - R\big]^2 = R^2, \tag{13.58}$$

where

$$\omega_c = \frac{qB}{m}, \quad v_q = \frac{E}{B} \quad \text{and} \quad R = \frac{v_q}{\omega_c}. \tag{13.59}$$

(b) Show that under a Lorentz transformation (for boost in x direction) the electric and magnetic fields transform as

$$\mathbf{E}' = \hat{\mathbf{y}}\, E', \tag{13.60a}$$

$$c\mathbf{B}' = \hat{\mathbf{z}}\, B'\eta(y), \tag{13.60b}$$

where

$$E' = \gamma(E - \beta cB), \tag{13.61a}$$

$$cB' = \gamma(cB - \beta E). \tag{13.61b}$$

Verify that

$$E'^2 - (cB)'^2 = E^2 - (cB)^2 \tag{13.62}$$

and

$$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E} \cdot \mathbf{B} = 0. \tag{13.63}$$

(c) Verify that for $\beta = \beta_d < 1$ we have $B' = 0$ and $E' = E/\gamma_d$. Investigate what happens to the radius R and the pitch of the cycloid $2\pi R$ in this case.

(d) Note that for $\beta = E/(cB) > 1$ we have $B' = B/\gamma$ and $E' = 0$. Investigate what happens.

Chapter 14

Electrodynamics of moving bodies

14.1 Retarded Green's function

1. (**20 points.**) Using Maxwell's equations, without introducing potentials, show that the electric and magnetic fields satisfy the inhomogeneous wave equations

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = -\frac{1}{\varepsilon_0}\boldsymbol{\nabla}\rho(\mathbf{r},t) - \frac{1}{\varepsilon_0}\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{J}(\mathbf{r},t), \tag{14.1a}$$

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0\boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{14.1b}$$

2. (**20 points.**) From Maxwell's equations, including magnetic charges and currents,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}\rho_e, \qquad -\boldsymbol{\nabla}\times\mathbf{E} - \mu_0\frac{\partial\mathbf{H}}{\partial t} = \mathbf{J}_m, \tag{14.2a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{H} = \frac{1}{\mu_0}\rho_m, \qquad \boldsymbol{\nabla}\times\mathbf{H} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mathbf{J}_e, \tag{14.2b}$$

derive the inhomogeneous wave equation

$$\left(-\nabla^2 + \varepsilon_0\mu_0\frac{\partial^2}{\partial t^2}\right)\mathbf{H} = -\frac{1}{\mu_0}\boldsymbol{\nabla}\rho_m - \varepsilon_0\frac{\partial}{\partial t}\mathbf{J}_m + \boldsymbol{\nabla}\times\mathbf{J}_e. \tag{14.3a}$$

3. (**70 points.**) The n-dimensional Euclidean Green's function satisfies

$$-\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)G_E^{(n)}(x_1,\ldots,x_n) = \delta(x_1)\cdots\delta(x_n). \tag{14.4}$$

(a) Show that the solution to this equation can be written as the Fourier transform

$$G_E^{(n)}(x_1,\ldots,x_n) = \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dk_n}{2\pi}\,\frac{e^{i(k_1x_1+\ldots+k_nx_n)}}{k_1^2+\ldots+k_n^2}. \tag{14.5}$$

(b) Verify the integral

$$\frac{1}{M} = \int_0^\infty ds\, e^{-sM}. \tag{14.6}$$

(c) Using Eq. (14.6) in Eq. (14.5) show that

$$G_E^{(n)}(x_1,\ldots,x_n) = \int_0^\infty ds\prod_{m=1}^{n}\left[\int_{-\infty}^{\infty}\frac{dk_m}{2\pi}e^{-sk_m^2+ik_mx_m}\right]. \tag{14.7}$$

(d) Show that

$$\int_{-\infty}^{\infty} dk_m e^{-sk_m^2+ik_mx_m} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{x_m^2}{4s}} \tag{14.8}$$

(e) Substitute the integral of Eq. (14.8) in Eq. (14.7), and use the integral representation of Gamma function,

$$\Gamma(z) = \int_0^\infty \frac{dt}{t} t^z e^{-t}, \tag{14.9}$$

where $\Gamma(z)$ is the analytic continuation of factorial, $n! = \Gamma(n+1)$, after substituting $s = 1/t$ there, to show that

$$G_E^{(n)}(x_1,\ldots,x_n) = \left(\frac{\sqrt{\pi}}{2\pi}\right)^n \Gamma\left(\frac{n}{2}-1\right)\left(\frac{4}{x_1^2+\ldots+x_n^2}\right)^{\frac{n}{2}-1}. \tag{14.10}$$

(f) Verify that

$$G_E^{(3)} = \frac{1}{4\pi}\frac{1}{R_3} \tag{14.11}$$

and

$$G_E^{(4)} = \frac{1}{4\pi^2}\frac{1}{R_4^2}, \tag{14.12}$$

where $R_n^2 = x_1^2 + \ldots + x_n^2$.

(g) Show that integration of the Euclidean Green's function over one coordinate leads to the Euclidean Green's function in one lower dimension,

$$\int_{-\infty}^{\infty} dx_n G_E^{(n)}(x_1,\ldots,x_n) = G_E^{(n-1)}(x_1,\ldots,x_{n-1}). \tag{14.13}$$

Hint: Substitute $x_n = R_{n-1}\tan\theta$ and use the integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta (\cos\theta)^{n-4} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{n}{2}-\frac{3}{2}\right)}{\Gamma\left(\frac{n}{2}-1\right)}, \qquad \mathrm{Re}\, n > 3. \tag{14.14}$$

4. (**80 points.**) The 4-dimensional Euclidean Green's function satisfies

$$-\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_4^2}\right) G_E(x_1,\ldots,x_4) = \delta(x_1)\cdots\delta(x_4). \tag{14.15}$$

(a) Show that the solution to this equation can be written as the Fourier transform

$$G_E(x_1,\ldots,x_4) = \int_{-\infty}^{\infty}\frac{dk_1}{2\pi}\cdots\int_{-\infty}^{\infty}\frac{dk_4}{2\pi}\,\frac{e^{i(k_1x_1+\ldots+k_4x_4)}}{k_1^2+\ldots+k_4^2}. \tag{14.16}$$

(b) Verify the integral

$$\frac{1}{M} = \int_0^\infty ds\, e^{-sM}. \tag{14.17}$$

(c) Using Eq. (14.17) in Eq. (14.16) show that

$$G_E(x_1,\ldots,x_4) = \int_0^\infty ds \prod_{m=1}^{4}\left[\int_{-\infty}^{\infty}\frac{dk_m}{2\pi} e^{-sk_m^2+ik_mx_m}\right]. \tag{14.18}$$

(d) Show that

$$\int_{-\infty}^{\infty} dk_m e^{-sk_m^2+ik_mx_m} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{x_m^2}{4s}} \tag{14.19}$$

(e) Using the integral of Eq. (14.19) in Eq. (14.18) and using the integral representation of Gamma function,

$$\Gamma(z) = \int_0^\infty \frac{ds}{s} s^z e^{-s}, \tag{14.20}$$

show that

$$G_E(x_1, \ldots, x_4) = \frac{1}{4\pi^2} \frac{1}{x_1^2 + \ldots + x_4^2}. \tag{14.21}$$

(f) By making the complex replacement

$$x_4 \to ict \equiv \lim_{\varepsilon \to 0+} e^{i(\frac{\pi}{2} - \varepsilon)} ct, \tag{14.22}$$

and defining

$$D_+(\mathbf{r}, t) = iG_E(\mathbf{r}, ict) \tag{14.23}$$

deduce the following differential equation for $D_+(\mathbf{r}, t)$:

$$\left(-\nabla^2 + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) D_+(\mathbf{r}, t) = \delta^{(3)}(\mathbf{r}) \delta(ct), \tag{14.24}$$

where $\mathbf{r} = (x_1, x_2, x_3)$, with the corresponding solution

$$D_+(\mathbf{r}, t) = \lim_{\varepsilon \to 0+} \frac{i}{4\pi^2} \frac{1}{\mathbf{r}^2 - (ct)^2 + i\varepsilon'}, \tag{14.25}$$

where $\varepsilon' = (ct)^2 \varepsilon$.

(g) Using the δ-function representation

$$\pi \delta(x) = \lim_{\varepsilon \to 0+} \frac{\varepsilon}{x^2 + \varepsilon^2} = \lim_{\varepsilon \to 0+} \mathrm{Im} \frac{1}{x - i\varepsilon} \tag{14.26}$$

and the identity

$$\delta(r^2 - (ct)^2) = \frac{1}{2r} \left[\delta(r - ct) + \delta(r + ct) \right] \tag{14.27}$$

show that

$$\mathrm{Re}\, D_+(\mathbf{r}, t) = \frac{1}{2} \left[\frac{\delta(r - ct)}{4\pi r} + \frac{\delta(r + ct)}{4\pi r} \right], \tag{14.28}$$

where the two terms here are the retarded and advanced Green's functions, respectively, up to numerical factors.

(h) Refer problem 31.9 in Schwinger et al. for further discussion on this subject. (Will not be graded.)

5. (**20 points.**) Consider the retarded Green's function

$$G(\mathbf{r} - \mathbf{r}', t - t') = \frac{1}{4\pi |\mathbf{r} - \mathbf{r}'|} \delta \left(t - t' - \frac{1}{c} |\mathbf{r} - \mathbf{r}'| \right). \tag{14.29}$$

(a) For $\mathbf{r}' = 0$ and $t' = 0$ show that

$$G(r, t) = \frac{1}{4\pi r} \delta \left(t - \frac{r}{c} \right). \tag{14.30}$$

(b) Then, evaluate

$$\int_{-\infty}^{\infty} dt\, G(r, t). \tag{14.31}$$

(c) From the answer above, what can you comment on the physical interpretation of $\int_{-\infty}^{\infty} dt\, G(r, t)$.

6. (**20 points.**) The 4-dimensional Euclidean Green's function satisfies

$$-\left(\boldsymbol{\nabla}^2 + \frac{\partial^2}{\partial x_4^2}\right) G_E(\mathbf{r}, x_4) = \delta^{(3)}(\mathbf{r})\delta(x_4) \tag{14.32}$$

and has the solution

$$G_E(\mathbf{r}, x_4) = \frac{1}{4\pi^2}\frac{1}{\mathbf{r}^2 + x_4^2}. \tag{14.33}$$

Evaluate the integral

$$\int_{-\infty}^{\infty} dx_4\, G_E(\mathbf{r}, x_4). \tag{14.34}$$

From the answer what can you comment about the physical interpretation of $\int_{-\infty}^{\infty} dx_4\, G_E$?

14.2 Retarded time

1. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, g(x)\, \delta(b^2 - a^2x^2). \tag{14.35}$$

Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{14.36}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.

2. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{14.37}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, determine the associated identity for

$$\delta(ax^2 + bx + c). \tag{14.38}$$

3. (**20 points.**) Evalauate the integral

$$\int_{-\infty}^{\infty} dx\, e^{ix}\, \delta(x^2 - a^2) \tag{14.39}$$

for $a > 0$. Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{14.40}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$.

4. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x - a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{14.41}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, determine the associated identity for

$$\delta(x^3 - 6x^2 + 11x - 6). \tag{14.42}$$

5. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\big|_{x=a_r}\right|}, \tag{14.43}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, evaluate the integral (requiring the roots to be causal, that is, $t_r < t$)

$$\int_{-\infty}^{\infty} dt' \frac{\delta\left(t - t' - \frac{1}{c}\sqrt{x^2+y^2+(z-vt')^2}\right)}{\sqrt{x^2+y^2+(z-vt')^2}}. \tag{14.44}$$

6. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\big|_{x=a_r}\right|}, \tag{14.45}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$, determine the associated identities for

$$\delta(\sin x), \quad \delta(\cos x), \quad \text{and} \quad \delta(\tan x). \tag{14.46}$$

7. (**20 points.**) Evaluate the integral

$$\lim_{\epsilon\to 0+} \int_{0-\epsilon}^{\infty} dx \, \frac{x^{\frac{x}{\pi}}\,\delta(\sin x)}{\Gamma\left(\frac{x}{\pi}+1\right)} \tag{14.47}$$

as a sum. Recognize the sum as an elementary function. Here $\Gamma(n+1) = n!$.

8. (**20 points.**) Evaluate the integral

$$\int_{-\infty}^{\infty} dx \, e^{-x^2}\,\delta(\sin x) \tag{14.48}$$

as a sum. The resultant sum is the elliptic theta function.
Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\big|_{x=a_r}\right|}, \tag{14.49}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$.

9. (**20 points.**) Evaluate the integral

$$\zeta(s) = \lim_{\epsilon\to 0+} \int_{\epsilon}^{\infty} dx \left(\frac{\pi}{x}\right)^s \delta(\sin x) \tag{14.50}$$

as a sum. The resultant sum is the Riemann zeta function. Determine $\zeta(2)$.
Hint: Use the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\big|_{x=a_r}\right|}, \tag{14.51}$$

where the sum on r runs over the roots a_r of the equation $F(x)=0$.

14.3 A charged particle moving with uniform velocity

1. (**20 points.**) A particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$, such that the position of the particle is

$$\mathbf{r}(t) = 0\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} + vt\,\hat{\mathbf{k}}. \tag{14.52}$$

The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r},t) = \gamma\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}, \tag{14.53a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta\gamma\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}. \tag{14.53b}$$

Using a clear diagram illustrate the direction of the fields at position (x,y,z) relative to the position of the particle at time t.

2. (**20 points.**) A charged particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$. The scalar and vector potential generated by this charged particle is

$$\phi(\mathbf{r},t) = \gamma\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{(x^2+y^2)+\gamma^2(z-vt)^2}}, \tag{14.54a}$$

$$c\mathbf{A}(\mathbf{r},t) = \beta\gamma\frac{q}{4\pi\varepsilon_0}\frac{\hat{\mathbf{z}}}{\sqrt{(x^2+y^2)+\gamma^2(z-vt)^2}}. \tag{14.54b}$$

(a) Using

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial}{\partial t}\mathbf{A}, \tag{14.55a}$$

$$\mathbf{A} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{14.55b}$$

evaluate the electric and magnetic field generated by the charged particle to be

$$\mathbf{E}(\mathbf{r},t) = \gamma\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}, \tag{14.56a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta\gamma\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}. \tag{14.56b}$$

(b) Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r},t) = \varepsilon_0\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t). \tag{14.57}$$

3. (**20 points.**) A charge particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$. The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r},t) = \gamma\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}, \tag{14.58a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta\gamma\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}. \tag{14.58b}$$

Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r},t) = \varepsilon_0\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t) \tag{14.59}$$

and the flux of electromagnetic energy density for this configuration by evaluating

$$\mathbf{S}(\mathbf{r},t) = \varepsilon_0 c^2\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t). \tag{14.60}$$

To do:

(a) Determine the ratio $\hat{\mathbf{z}}\cdot\mathbf{S}$ and $\hat{\boldsymbol{\rho}}\cdot\mathbf{S}$. Interpret.

4. (**20 points.**) A charge particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$. The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r},t) = \gamma\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}, \tag{14.61a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta\gamma\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}}+x\hat{\mathbf{j}}}{[(x^2+y^2)+\gamma^2(z-vt)^2]^{\frac{3}{2}}}. \tag{14.61b}$$

Evaluate the electromagnetic field invariants,

$$\mathbf{E}(\mathbf{r},t)^2 - c^2\mathbf{B}(\mathbf{r},t)^2 = \left(\frac{q}{4\pi\varepsilon_0}\frac{1}{[(x^2+y^2)+\gamma^2(z-vt)^2]}\right)^2 \tag{14.62}$$

and

$$\mathbf{E}(\mathbf{r},t)\cdot c\mathbf{B}(\mathbf{r},t) = 0. \tag{14.63}$$

Verify that

$$\mathbf{E}'(\mathbf{r}',t')^2 - c^2\mathbf{B}'(\mathbf{r}',t')^2 = \mathbf{E}(\mathbf{r},t)^2 - c^2\mathbf{B}(\mathbf{r},t)^2 \tag{14.64}$$

and

$$\mathbf{E}'(\mathbf{r}',t')\cdot\mathbf{B}'(\mathbf{r}',t') = \mathbf{E}(\mathbf{r},t)\cdot\mathbf{B}(\mathbf{r},t). \tag{14.65}$$

5. (**40 points.**) Consider a particle of charge q moving along the path $\mathbf{r}_q(t)$. The corresponding charge density and current density are

$$\rho(\mathbf{r}',t') = q\,\delta^{(3)}(\mathbf{r}'-\mathbf{r}_q(t')), \tag{14.66a}$$

$$\mathbf{j}(\mathbf{r}',t') = q\,\mathbf{v}_q(t')\delta^{(3)}(\mathbf{r}'-\mathbf{r}_q(t')), \tag{14.66b}$$

where $\mathbf{v}_q(t)$ is the velocity of the particle at time t.

(a) Beginning from

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\int_{-\infty}^{\infty}dt'\,\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right), \tag{14.67a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\int_{-\infty}^{\infty}dt'\,\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right), \tag{14.67b}$$

and using Eqs. (14.66) derive

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}dt'\,\frac{\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}_q(t')|\right)}{|\mathbf{r}-\mathbf{r}_q(t')|}, \tag{14.68a}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dt'\,q\mathbf{v}_q(t')\,\frac{\delta\left(t-t'-\frac{1}{c}|\mathbf{r}-\mathbf{r}_q(t')|\right)}{|\mathbf{r}-\mathbf{r}_q(t')|}. \tag{14.68b}$$

(b) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\left.\frac{dF}{dx}\right|_{x=a_r}\right|}, \tag{14.69}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate the integrals (requiring the roots to be causal, that is, $t_r < t$) in Eqs. (14.68) as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{\left[|\mathbf{r} - \mathbf{r}_q(t_r)| - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{ \mathbf{r} - \mathbf{r}_q(t_r) \Big\} \right]}, \tag{14.70a}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v}_q(t')}{\left[|\mathbf{r} - \mathbf{r}_q(t_r)| - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{ \mathbf{r} - \mathbf{r}_q(t_r) \Big\} \right]}, \tag{14.70b}$$

where t_r is uniquely determined using

$$F(t_r) = c(t - t_r) - |\mathbf{r} - \mathbf{r}_q(t_r)| = 0, \qquad t_r < t. \tag{14.71}$$

(c) In terms of the four-vectors

$$x^\alpha - x_q^\alpha(t_r) = (ct - ct_r, \mathbf{r} - \mathbf{r}_q(t_r)) \tag{14.72}$$

and

$$u_q^\alpha = \gamma_q(c, \mathbf{v}_q(t_r)), \qquad \gamma_q = \frac{1}{\sqrt{1 - \frac{\mathbf{v}_q(t_r)^2}{c^2}}}, \tag{14.73}$$

show that the expression in the denominator can be interpreted as

$$-\frac{1}{c\gamma_q}(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = c(t - t_r) - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{ \mathbf{r} - \mathbf{r}_q(t_r) \Big\} \tag{14.74a}$$

$$= |\mathbf{r} - \mathbf{r}_q(t_r)| - \frac{\mathbf{v}_q(t_r)}{c} \cdot \Big\{ \mathbf{r} - \mathbf{r}_q(t_r) \Big\}. \tag{14.74b}$$

CHECK: Thus, $F(t_r) = 0$ implies

$$(u_q)_\alpha(x^\alpha - x_q^\alpha(t_r)) = 0, \tag{14.75}$$

stating that these events are separated by light-like distance.

14.3.1 A charged particle moving with speed of light

1. (**20 points.**) The electric and magnetic field generated by a particle with charge q moving along the z axis with speed v, $\beta = v/c$, can be expressed in the form

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\big[x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z - vt)\hat{\mathbf{k}}\big]}{(x^2 + y^2)} \frac{(x^2 + y^2)(1 - \beta^2)}{[(x^2 + y^2)(1 - \beta^2) + (z - vt)^2]^{\frac{3}{2}}}, \tag{14.76a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \boldsymbol{\beta} \times \mathbf{E}(\mathbf{r}, t). \tag{14.76b}$$

(a) Consider the distribution

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{2} \frac{\epsilon}{(x^2 + \epsilon)^{\frac{3}{2}}}. \tag{14.77}$$

Show that

$$\delta(x) \begin{cases} \to \frac{1}{2\sqrt{\epsilon}} \to \infty, & \text{if} \quad x = 0, \\ \to \frac{\epsilon}{2x^3} \to 0, & \text{if} \quad x \neq 0. \end{cases} \tag{14.78}$$

Further, show that

$$\int_{-\infty}^{\infty} dx\, \delta(x) = 1. \tag{14.79}$$

(b) Thus, verify that the electric and magnetic field of a charge approaching the speed of light can be expressed in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z-ct), \tag{14.80a}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct) = 2q\left(\frac{\mu_0 c}{4\pi}\right)\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct), \tag{14.80b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\rho = \sqrt{x^2+y^2}$, $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$, and $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ are the associated unit vectors. These fields are confined on the $z = ct$ plane moving with speed c. Illustrate this configuration of fields using a diagram.

(c) To confirm that the above confined fields are indeed solutions to the Maxwell equations, verify the following:

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\varepsilon_0}q\delta^{(2)}(\boldsymbol{\rho})\delta(z-ct), \tag{14.81a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{14.81b}$$

$$\boldsymbol{\nabla}\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{14.81c}$$

$$\boldsymbol{\nabla}\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} = \mu_0 qc\hat{\mathbf{z}}\delta^{(2)}(\boldsymbol{\rho})\delta(z-ct). \tag{14.81d}$$

This is facilitated by writing

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{14.82}$$

and accomplished by using the following identities:

$$\boldsymbol{\nabla}_\rho\cdot\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 2\pi\delta^{(2)}(\boldsymbol{\rho}), \qquad \boldsymbol{\nabla}_\rho\times\left(\frac{\hat{\boldsymbol{\rho}}}{\rho}\right) = 0, \tag{14.83a}$$

$$\boldsymbol{\nabla}_\rho\cdot\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = 0, \qquad \boldsymbol{\nabla}_\rho\times\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \hat{\mathbf{z}}\,2\pi\delta^{(2)}(\boldsymbol{\rho}). \tag{14.83b}$$

2. (**20 points.**) For a particle of charge q moving very close to the speed of light, $\beta \to 1$, we have the electric and magnetic fields as

$$\mathbf{E}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\rho}}}{\rho}\,\delta(z-ct), \tag{14.84a}$$

$$c\mathbf{B}(\mathbf{r},t) = \frac{2q}{4\pi\varepsilon_0}\frac{\hat{\boldsymbol{\phi}}}{\rho}\,\delta(z-ct), \tag{14.84b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. These fields are confined on a plane perpendicular to direction of motion.

(a) In the limit $\beta \to 1$ we have

$$\boldsymbol{\beta} = \beta\,\hat{\mathbf{z}} \to \hat{\boldsymbol{\beta}}. \tag{14.85}$$

Show that

$$c\mathbf{B}(\mathbf{r},t) = \hat{\boldsymbol{\beta}}\times\mathbf{E}(\mathbf{r},t). \tag{14.86}$$

(b) The electromagnetic energy density involves bilinear constructions of fields. When the fields are confined to a plane, these involve bilinear δ-functions that needs to be carefully interpreted. In particular, we encounter

$$\delta(z-vt)\delta(z-vt) = \delta(z-vt)\delta(0), \tag{14.87}$$

where $\delta(0)$ is interpreted as the inverse of the infinitely small length L_z associated to the plane on which the fields are confined. That is,

$$\delta(0) = \lim_{L_z \to 0} \frac{1}{L_z}. \tag{14.88}$$

Starting from

$$U_e(\mathbf{r}, t) = \frac{\varepsilon_0}{2} E^2 \tag{14.89}$$

show that the contribution to the energy density from the electric field can be expressed in the form

$$\frac{\text{Energy}}{\text{Area}} = \lim_{L_z \to 0} U_e(\mathbf{r}, t) L_z = \frac{1}{2\pi} \frac{q^2}{4\pi\varepsilon_0} \frac{1}{\rho^2} \delta(z - ct). \tag{14.90}$$

Similarly, starting from

$$U_m(\mathbf{r}, t) = \frac{1}{2\mu_0} B^2 \tag{14.91}$$

show that the energy density from the magnetic field is given by

$$\frac{\text{Energy}}{\text{Area}} = \lim_{L_z \to 0} U_m(\mathbf{r}, t) L_z = \frac{1}{2\pi} \frac{q^2}{4\pi\varepsilon_0} \frac{1}{\rho^2} \delta(z - ct). \tag{14.92}$$

Thus, show that the ratio of electric to magnetic energy density,

$$\frac{U_m(\mathbf{r}, t)}{U_e(\mathbf{r}, t)} = 1 \tag{14.93}$$

for the above configuration.

(c) Evaluate

$$\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t) \tag{14.94}$$

for the above configuration. Show that

$$\mathbf{G} = \varepsilon_0 \mathbf{E} \times \mathbf{B} = \hat{\boldsymbol{\beta}} \frac{U}{c}, \tag{14.95}$$

where $U = U_e + U_m$.

(d) A plane wave is characterized by $U_e = U_m$ and $\mathbf{E} \cdot \mathbf{B} = 0$. Does the above configuration satisfy the characteristics of a plane wave?

14.3.2 A point electric dipole moving with uniform velocity

1. (**20 points.**)
REWRITE this question. The limits leads to a δ-function.
Consider a point electric dipole moment $\mathbf{d}$ moving with velocity $\mathbf{v} = v\hat{\mathbf{z}}$. For the case of time independent $\mathbf{d}$ and $\mathbf{v}$, and when the dipole moves close to speed of light, $\beta \to 1$, we can write the leading order contributions in $(1 - \beta^2)$ for the electric and magnetic fields as

$$\mathbf{E}(\mathbf{r}, t) = \begin{cases} \dfrac{1}{\sqrt{1-\beta^2}} \dfrac{1}{4\pi\varepsilon_0} (-\mathbf{d} \cdot \boldsymbol{\nabla}) \dfrac{\boldsymbol{\rho}}{\rho^3}, & z = vt, \\ 0, & z \neq vt, \end{cases} \tag{14.96a}$$

$$c\mathbf{B}(\mathbf{r}, t) = \begin{cases} \dfrac{\beta}{\sqrt{1-\beta^2}} \dfrac{1}{4\pi\varepsilon_0} (-\mathbf{d} \cdot \boldsymbol{\nabla}) \dfrac{\boldsymbol{\phi}}{\rho^3}, & z = vt, \\ 0, & z \neq vt, \end{cases} \tag{14.96b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. These fields are confined on a plane perpendicular to direction of motion. Determine the electromagnetic momentum density flux for the particular configuration $\mathbf{d} = d\hat{\boldsymbol{\rho}}$ by calculating

$$\mathbf{E} \times \mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E} \times c\mathbf{B}. \tag{14.97}$$

14.3.3 A charged particle undergoing hyperbolic motion

1. (**20 points.**) The following problem is a challenge problem. Find the fields for a charged particle with charge q undergoing hyperbolic motion while moving on the z-axis, described by

$$\mathbf{r}_q(t) = \hat{\mathbf{x}}\,0 + \hat{\mathbf{y}}\,0 + \hat{\mathbf{z}}\sqrt{c^2t^2 + z_0^2}. \tag{14.98}$$

Study the article by Franklin and Griffiths (arXiv:1405.7729). Try to reproduce the results there as much as you can. Do a forward literature search, that is, find the articles referring back to this article. Summarize the latest status of this conundrum.

Chapter 15

Electromagnetic radiation

15.1 Radiation: time domain

1. (**20 points.**) Given the retarded time (in the far-field approximation)

$$t_r = t - \frac{r}{c} + \frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}, \tag{15.1}$$

evaluate

$$\boldsymbol{\nabla} t_r \tag{15.2}$$

and

$$\boldsymbol{\nabla}' t_r. \tag{15.3}$$

2. (**20 points.**) The electromagnetic fields,

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{15.4a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{15.4b}$$

in the Maxwell equations, in SI units,

$$\boldsymbol{\nabla}\cdot\mathbf{D} = \rho, \tag{15.5a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 0, \tag{15.5b}$$

$$-\boldsymbol{\nabla}\times\mathbf{E} - \frac{\partial \mathbf{B}}{\partial t} = 0, \tag{15.5c}$$

$$\boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \tag{15.5d}$$

are determined in terms of the electric scalar potential ϕ and the magnetic vector potential $\mathbf{A}$ by the relations

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} \qquad \text{and} \qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{15.6}$$

These potentials are not uniquely defined, because if we let

$$\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda, \qquad \phi \to \phi - \frac{\partial \lambda}{\partial t}, \tag{15.7}$$

the electric and magnetic fields in Eq. (15.6) remain unaltered for an arbitrary function $\lambda = \lambda(\mathbf{r}, t)$. This is called gauge invariance or gauge symmetry. This symmetry allows us to choose a gauge for simplifying a calculation. In the Lorenz gauge,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0, \tag{15.8}$$

the electric scalar potential and the magnetic vector potential are given in terms of inhomogeneous wave equations,

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\mathbf{r},t) \;=\; \frac{1}{\varepsilon_0}\rho(\mathbf{r},t), \tag{15.9a}$$

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) \;=\; \mu_0\mathbf{J}(\mathbf{r},t). \tag{15.9b}$$

The associated Green function defined using the differential equation

$$-\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}-\mathbf{r}',t-t') = \delta^{(3)}(\mathbf{r}-\mathbf{r}')\delta(t-t') \tag{15.10}$$

has solution

$$G(\mathbf{r}-\mathbf{r}',t-t') = \frac{\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{15.11}$$

This resembles the electric potential due to a unit point charge in electrostatics, however, it now accounts for dynamics, primarily as retardation in time.

(a) Show that the electric scalar potential and the magnetic vector potential, after completing the integral on t', are formally determined in terms of the following integrals,

$$\phi(\mathbf{r},t) \;=\; \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}, \tag{15.12a}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}\left(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}. \tag{15.12b}$$

(b) The non-retarded limit corresponds to the approximation

$$|\mathbf{r}-\mathbf{r}'| \ll c(t-t'). \tag{15.13}$$

Here $\mathbf{r}'$ and t', even though they are integral parameters, are physical, because they are associated to the distribution of sources. The non-retarded limit is consistent with assuming $c\to\infty$, the non-relativistic limit. Show that in this limit we have

$$\phi(\mathbf{r},t) \;=\; \frac{1}{4\pi\varepsilon_0}\int d^3r'\,\frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}, \tag{15.14a}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{15.14b}$$

Though the resemblance is striking note that this is not still the static limit, because the time dependence in the magnetic vector potential contributes to the electric field.

(c) Radiation fields corresponds to the approximation

$$|\mathbf{r}-\mathbf{r}'| \gg c(t-t'). \tag{15.15}$$

Show that in this far-field limit

$$|\mathbf{r}-\mathbf{r}'| = r\left(1-\hat{\mathbf{r}}\cdot\frac{\mathbf{r}'}{r}\right)+\mathcal{O}\left(\frac{r'}{r}\right)^2. \tag{15.16}$$

Show that in the far-field approximation

$$\phi(\mathbf{r},t) \;=\; \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int d^3r'\rho\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right), \tag{15.17a}$$

$$\mathbf{A}(\mathbf{r},t) \;=\; \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\mathbf{J}\left(\mathbf{r}',t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}\right). \tag{15.17b}$$

(d) Define the retarded time

$$t_r = t - \frac{r}{c} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}. \tag{15.18}$$

Show that

$$\boldsymbol{\nabla} t_r = -\frac{\hat{\mathbf{r}}}{c} - \frac{1}{c}\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{\mathbf{r}'}{r}\right) = -\frac{\hat{\mathbf{r}}}{c} + \mathcal{O}\left(\frac{r'}{r}\right). \tag{15.19}$$

(e) Show that the leading contributions are

$$\boldsymbol{\nabla}\phi(\mathbf{r}, t) = -\frac{1}{c}\frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}}{r}\int d^3r' \left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.20a}$$

$$\frac{\partial}{\partial t}\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.20b}$$

$$\boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r}, t) = -\frac{1}{c}\frac{\mu_0}{4\pi}\frac{\hat{\mathbf{r}}}{r} \times \int d^3r' \left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.20c}$$

Thus, derive

$$c\mathbf{B}(\mathbf{r}, t) = -\hat{\mathbf{r}} \times \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.21a}$$

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \left[\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r} - c\hat{\mathbf{r}}\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}', t')\right\}_{t'=t_r}\right]. \tag{15.21b}$$

(f) Recall that Maxwell's equations implies the local charge conservation,

$$\frac{\partial}{\partial t'}\rho(\mathbf{r}', t') + \boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}', t') = 0. \tag{15.22}$$

Thus, we have

$$\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}', t')\right\}_{t'=t_r} = -\left\{\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r=t-\frac{r}{c}+\frac{\hat{\mathbf{r}}\cdot\mathbf{r}'}{c}}, \tag{15.23}$$

where we emphasize that the substitution $t' = t_r$ is made after completing the divergence with respect to $\mathbf{r}'$. By reversing this order show that

$$\left\{\frac{\partial}{\partial t'}\rho(\mathbf{r}', t')\right\}_{t'=t_r} = -\boldsymbol{\nabla}' \cdot \mathbf{J}(\mathbf{r}', t_r) + \frac{\hat{\mathbf{r}}}{c} \cdot \left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}. \tag{15.24}$$

Under integration with respect to $\mathbf{r}'$ the first term on the right hand side contributes only on the surface. Thus, argue that this term does not contribute in the far-field zone. Then, recognize the vector triple product to deduce

$$\mathbf{E}(\mathbf{r}, t) = \hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \times \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}\right). \tag{15.25}$$

(g) Verify that these fields satisfy

$$\mathbf{E} = -\hat{\mathbf{r}} \times c\mathbf{B}, \tag{15.26a}$$

$$c\mathbf{B} = \hat{\mathbf{r}} \times \mathbf{E}. \tag{15.26b}$$

Thus, $\hat{\mathbf{r}}$, $\mathbf{E}$, and $\mathbf{B}$, are orthogonal to each other. Further, we have $c^2B^2 = E^2$, which can be rewritten in the form

$$\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2, \tag{15.27}$$

which states that the energy stored in the radiation field is equally divided in the electric and magnetic fields. Recall that plane monochromatic waves also satisfied these properties.

3. (**20 points.**) Neglecting quadruple and higher moments, the angular distribution of power radiated by a non-relativistic particle is given by

$$\frac{dP}{d\Omega} = \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\frac{1}{c^2}\big[(\hat{\mathbf{r}}\times\ddot{\mathbf{d}})^2 + (\hat{\mathbf{r}}\times\ddot{\boldsymbol{\mu}})^2 + 2\hat{\mathbf{r}}\cdot(\ddot{\mathbf{d}}\times\ddot{\boldsymbol{\mu}})\big]. \tag{15.28}$$

Calculate the contribution to the total power radiated $P(t)$ from the third term, that represents interference between $\mathbf{d}$ and $\boldsymbol{\mu}$, by integrating over all solid angles.

4. (**50 points.**) A particle, of charge q and mass m, always moves with speed $v \ll c$.

   (a) Consider the case when it oscillates on the x-axis with frequency ω_0 and amplitude A given by

   $$\mathbf{r}_1(t) = \hat{\mathbf{x}} A\cos\omega_0 t. \tag{15.29}$$

   Obtain expressions for the radiated electric field $\mathbf{E}(\mathbf{r},t)$, radiated magnetic field $\mathbf{B}(\mathbf{r},t)$, angular distribution of the radiated power $dP/d\Omega$, and the total power radiated P.

   (b) Next, consider the case when the particle moves on a circle described by

   $$\mathbf{r}_2(t) = \hat{\mathbf{x}} A\cos\omega_0 t + \hat{\mathbf{y}} A\sin\omega_0 t. \tag{15.30}$$

   Obtain expressions for the radiated electric field $\mathbf{E}(\mathbf{r},t)$, radiated magnetic field $\mathbf{B}(\mathbf{r},t)$, angular distribution of the radiated power $dP/d\Omega$, and the total power radiated P.

   (c) Show that the radiated electric and magnetic field is additive, that is, it is the sum of two oscillators.

   (d) Show that the radiated power is not additive, but exhibits interference effects. Identify the interference term for the circular motion.

   (e) Find directions $\hat{\mathbf{r}}$ for which the interference term goes to zero.

5. (**20 points.**) (Schwinger et al., problem 32.1.) A particle, of charge q and mass m, moves with speed $v \ll c$, in a uniform magnetic field $\mathbf{B}$. Suppose the motion is confined to the plane perpendicular to $\mathbf{B}$. Calculate the power radiated P in terms of B and v, and show that

$$P = -\frac{dE}{dt} = \gamma E, \tag{15.31}$$

where $E = mv^2/2$ is the energy of the particle. Find γ. Since then

$$E(t) = E(0)\, e^{-\gamma t}, \tag{15.32}$$

$1/\gamma$ is the mean lifetime of the motion. For an electron, find $1/\gamma$ in seconds for a magnetic field of 10^4 Gauss.

6. (**20 points.**) An electron of charge e and mass m moves in a nearly circular orbit under the Coulomb forces produced by a proton. Suppose, as it radiates, the electron continues to move on a circle of ever decreasing radii.

   (a) The equation of motion for the electron given by Newton's laws of motion is

   $$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}, \tag{15.33}$$

   where the acceleration of the election is the centripetal acceleration

   $$a = \frac{v^2}{r}. \tag{15.34}$$

   The total energy of the system E is the sum of the kinetic energy and electrostatic potential energy. Show that

   $$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \tag{15.35}$$

(b) A charge that is accelerating will loose energy in the form of radiation. The Larmor formula

$$P = -\frac{dE}{dt} = \left(\frac{\mu_0 c}{4\pi}\right)\frac{2}{3}\frac{e^2}{c^2}a^2, \tag{15.36}$$

gives the rate of loss of energy, the power P.

(c) Combine the equation of motion of the electron with the Larmor formula to contruct the following differential equation for the radius r,

$$\frac{1}{c}\frac{dr}{dt} = -\frac{4}{3}\frac{r_0^2}{r^2}, \tag{15.37}$$

where $r_0 \sim 3 \times 10^{-15}$ m is the classical radius of the electron defined using the equality

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_0} = mc^2, \tag{15.38}$$

that is, when the electrostatic interaction energy is sufficient to create an electron. Solve this differential equation. In a finite time the electron reaches the center. Calculate how long it takes for the electron to hit the proton if it starts from an initial radius $a_0 \sim 0.5 \times 10^{-10}$ m, the Bohr radius. This is the classical lifetime of a Bohr atom.

The following article by J. D. Olsen and K. T. McDonald titled 'Classical Lifetime of a Bohr Atom' available at

http://www.physics.princeton.edu/∼mcdonald/examples/orbitdecay.pdf

is recommended for reading.

(d) Most atoms have lifetimes greater than the age of the universe, which is about 10^{17} s. This instability was one of the reasons for the discovery of quantum mechanics.

7. Consider the motion of a non-relativisitic particle (speed v small compared to speed of light c, $v \ll c$,) of charge q and mass m. The charge oscillates on the x-axis with frequency ω_0 and amplitude A given by

$$\mathbf{r}_a(t) = \hat{\mathbf{i}}\, A\cos\omega_0 t. \tag{15.39}$$

(a) Find the acceleration of the particle

$$\mathbf{a}_a(t) = \frac{d^2}{dt^2}\mathbf{r}_a(t). \tag{15.40}$$

(b) Find the angular distribution of the radiated power

$$f(\theta,\phi,t) = \frac{dP}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{4\pi c^3}\big[\hat{\mathbf{r}} \times \mathbf{a}(t_e)\big]^2 \tag{15.41}$$

and the total radiated power

$$P(t) = \left(\frac{\mu_0 c}{4\pi}\right)\frac{2q^2}{3c^2}\mathbf{a}^2(t_e), \tag{15.42}$$

where $\mathbf{a}(t_e)$ is the acceleration of the particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{15.43}$$

8. (**20 points.**) Consider the motion of a non-relativisitic particle (speed v small compared to speed of light c, $v \ll c$,) of charge q and mass m. The charge moves on a circle described by

$$\mathbf{r}(t) = \hat{\mathbf{i}}\, A\cos\omega_0 t + \hat{\mathbf{j}}\, A\sin\omega_0 t. \tag{15.44}$$

Find the total radiated power

$$P(t) = \left(\frac{\mu_0 c}{4\pi}\right)\frac{2q^2}{3c^2}\mathbf{a}^2(t_e), \tag{15.45}$$

where $\mathbf{a}(t_e)$ is the acceleration of the particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{15.46}$$

If your eye (that can sense visible light) were to observe radiation coming off many such particles with different oscillation frequency ω_0, which color would the radiation be dominated in?

9. (**20 points.**) Consider the motion of three non-relativistic particles (speed v_i small compared to speed of light c, $v_i \ll c$,) of identical charges $q_i = q$ and identical masses $m_i = m$, $i = 1, 2, 3$. The radiated power by the individual particles are given by the expressions

$$P_i(t) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{2q^2}{3c^2} \mathbf{a}_i^2(t_e), \tag{15.47}$$

where $\mathbf{a}_i(t_e)$ is the acceleration of the i-th particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{15.48}$$

Let the total contribution to radiated power from the three particles together be denoted by the subscript $(1 + 2 + 3)$. Consider the motion of three particles moving on a circle with same uniform speed while remaining at the vertices of a equilateral triangle at each moment. Find the total radiated power $P_{(1+2+3)}(t)$. (Hint: The centripetal acceleration is in the radial direction.)

10. (**20 points.**) Consider the motion of two non-relativisitic particles (speed v_i small compared to speed of light c, $v_i \ll c$,) of identical charges $q_i = q$ and identical masses $m_i = m$, $i = 1, 2$. The individual radiation fields $\mathbf{B}_i(\mathbf{r}, t)$ and $\mathbf{E}_i(\mathbf{r}, t)$, the angular distribution of emitted power $f_i(\theta, \phi, t)$, and the total radiated power $P_i(t)$, are given by the expressions,

$$c\mathbf{B}_i(\mathbf{r}, t) = -\frac{\mu_0}{4\pi} \frac{q}{r} \hat{\mathbf{r}} \times \mathbf{a}_i(t_e), \tag{15.49a}$$

$$\mathbf{E}_i(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q}{r} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{a}_i(t_e)), \tag{15.49b}$$

$$f_i(\theta, \phi, t) = \frac{dP_i}{d\Omega} = \frac{1}{4\pi} \left(\frac{\mu_0 c}{4\pi}\right) \frac{q^2}{c^2} \left[\hat{\mathbf{r}} \times \mathbf{a}_i(t_e)\right]^2, \tag{15.49c}$$

$$P_i(t) = \left(\frac{\mu_0 c}{4\pi}\right) \frac{2q^2}{3c^2} \mathbf{a}_i^2(t_e), \tag{15.49d}$$

where $\mathbf{a}_i(t_e)$ is the acceleration of the i-th particle at the time of emission

$$t_e = t - \frac{r}{c}. \tag{15.50}$$

Let the total contribution to a physical quantity from the two particles together be denoted by the subscript $(1 + 2)$.

(a) Show that

$$\mathbf{B}_{(1+2)}(\mathbf{r}, t) = \mathbf{B}_1(\mathbf{r}, t) + \mathbf{B}_2(\mathbf{r}, t), \tag{15.51a}$$

$$\mathbf{E}_{(1+2)}(\mathbf{r}, t) = \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t), \tag{15.51b}$$

thus, conclude that radiation fields are additive.

(b) Show that, in general, the angular distribution of radiated power and total radiated power from the two particles together is not additive and exhibits interference,

$$f_{(1+2)}(\theta, \phi, t) = f_1(\theta, \phi, t) + f_2(\theta, \phi, t) + f_{12}(\theta, \phi, t), \tag{15.52}$$

where

$$f_{12}(\theta, \phi, t) = 2\frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\frac{q^2}{c^2}\big[\mathbf{a}_1(t_e)\cdot\mathbf{a}_2(t_e) - (\hat{\mathbf{r}}\cdot\mathbf{a}_1(t_e))(\hat{\mathbf{r}}\cdot\mathbf{a}_2(t_e))\big], \tag{15.53}$$

and

$$P_{(1+2)}(t) = P_1(t) + P_2(t) + P_{12}(t), \tag{15.54}$$

where

$$P_{12}(t) = 2\frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\frac{2q^2}{3c^2}\mathbf{a}_1(t_e)\cdot\mathbf{a}_2(t_e). \tag{15.55}$$

Observe that the interference effect in the total radiated power is totally destructive for the case $\mathbf{a}_1(t_e)\cdot\mathbf{a}_2(t_e) = 0$. For this case, the interference effect in the angular distribution of radiated power is not necessarily destructive.

(c) Consider the motion of two particles moving on a circle with same uniform speed while remaining diametrically opposite to each other at each moment. Find the total radiated power $P_{(1+2)}(t)$. (Hint: The centripetal acceleration is in the radial direction.)

(d) Consider the motion of three particles moving on a circle with same uniform speed while remaining at the vertices of a equilateral triangle at each moment. Find the total radiated power $P_{(1+2+3)}(t)$.

(e) Find $P_{(1+2+3+4)}(t)$ for four particles moving on a circle such that they are at the vertices of a square at each moment.

(f) Find $P_{(1+\ldots+N)}(t)$ for N particles moving on a circle such that they are at the vertices of a N-sided polygon at each moment. Answer is zero.

(g) The quadrupole contribution will not be zero.

11. (Schwinger et al., problem 32.2.) A non-relativistic particle of charge q and mass m moves in a Hooke's law potential (a linear oscillator) with natural frequency ω_0. Find P, the power radiated. Recall that for such motion, the time-averaged kinetic and potential energy satisfy

$$\bar{T} = \bar{V} = \frac{1}{2}E. \tag{15.56}$$

Show then that the power radiated, averaged over one cycle is

$$P = -\frac{dE}{dt} = \gamma E, \tag{15.57}$$

and find γ. Compute $1/\gamma$ in seconds, for electron, when ω_0 is $10^{15}\,\text{sec}^{-1}$ (a characteristic atomic frequency, corresponding to visible light).

15.1.1 Simple antenna

1. (**20 points.**) The magnetic field associated to radiation fields is given by

$$c\mathbf{B}(\mathbf{r}, t) = -\hat{\mathbf{r}}\times\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r'\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.58}$$

where the contribution to the field comes at the retarded time

$$t_r = t - \frac{r}{c} + \hat{\mathbf{r}}\cdot\frac{\mathbf{r}'}{c}. \tag{15.59}$$

The associated electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = -\hat{\mathbf{r}}\times c\mathbf{B}(\mathbf{r}, t), \tag{15.60}$$

and satisfies

$$c\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{r}}\times\mathbf{E}(\mathbf{r}, t). \tag{15.61}$$

Starting from the statement of conservation of electromagnetic energy density

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} = 0, \tag{15.62}$$

where the electromagnetic energy density

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2}\frac{B^2}{\mu_0}, \tag{15.63}$$

the flux of electromagnetic energy density (the Poynting vector)

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}, \tag{15.64}$$

$\mathbf{B} = \mu_0\mathbf{H}$; integrating over an infinitely large sphere centered about the sources; using divergence theorem to rewrite the second term; presuming the sources to be zero in the radiation zone; we deduce the power dP radiated into the solid angle $d\Omega$ to be

$$dP = \lim_{r\to\infty} r^2 d\Omega\,\hat{\mathbf{r}}\cdot\mathbf{S}. \tag{15.65}$$

(a) Using $\hat{\mathbf{r}}\cdot\mathbf{S} = \hat{\mathbf{r}}\cdot(\mathbf{E}\times\mathbf{H}) = (\hat{\mathbf{r}}\times\mathbf{E})\cdot\mathbf{H}$ show that this leads to the expression

$$\frac{\partial P}{\partial\Omega} = \lim_{r\to\infty}\frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\left|\frac{\mathbf{B}(\mathbf{r},t)}{\frac{\mu_0}{4\pi}\frac{1}{r}}\right|^2. \tag{15.66}$$

Verify that $B/\left(\frac{\mu_0}{4\pi}\frac{1}{r}\right)$ has the dimensions of current. Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{15.67}$$

has the dimensions of resistance. Quantum phenomena in electromagnetism is characterized by the Planck's constant h and the associated fine-structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}, \tag{15.68}$$

a dimensionless physical constant. Verify that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \alpha\frac{\hbar}{e^2} = 29.9792458\,\Omega. \tag{15.69}$$

(b) A simple antenna consists of an infinitely thin conductor of length L carrying a time-dependent current. Let the conductor be centered at the origin and placed on the z axis such that

$$\mathbf{J}(\mathbf{r}',t') = \hat{\mathbf{z}}\,I_0\sin\omega_0 t'\,\delta(x')\delta(y')\theta(-L < 2z' < L). \tag{15.70}$$

The function θ equals 1 when it's argument is a true statement, and zero otherwise. Show that

$$\int d^3r'\left\{\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}',t')\right\}_{t'=t_r} = \hat{\mathbf{z}}\,\omega_0 I_0\cos\left(\omega_0 t - 2\pi\frac{r}{\lambda_0}\right)\frac{\sin\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\frac{\pi}{\lambda_0}\cos\theta}, \tag{15.71}$$

where $\omega_0/c = 2\pi/\lambda_0$. Then, evaluate the expression for the magnetic field.

(c) Using Eq. (15.66) show that

$$\frac{\partial P}{\partial\Omega} = P_0\frac{\sin^2\theta}{\pi}\cos^2\left(\omega_0 t - 2\pi\frac{r}{\lambda_0}\right)\frac{\sin^2\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta}, \tag{15.72}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \tag{15.73}$$

Evaluate the average power radiated into a solid angle using

$$\left\langle \frac{\partial P}{\partial \Omega} \right\rangle = \frac{1}{T}\int_0^T dt\, \frac{\partial P}{\partial \Omega}. \tag{15.74}$$

Show that

$$\left\langle \frac{\partial P}{\partial \Omega} \right\rangle = P_0 \frac{\sin^2\theta}{2\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta}. \tag{15.75}$$

Hint: Use the integral

$$\frac{1}{T}\int_0^T dt\, \cos^2(\omega_0 t + \delta) = \frac{1}{2}. \tag{15.76}$$

(d) Plot

$$g(\theta) = \sin^2\theta \frac{\sin^2\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} \tag{15.77}$$

as a function of θ for $L = 0.1\lambda, 0.5\lambda, 1.0\lambda, 2.0\lambda, 3.0\lambda, 5.0\lambda$. Thus, discuss the angular distribution of the radiation. Note that the radiated power is zero when

$$\theta = \cos^{-1}\left(n\frac{\lambda_0}{L}\right), \qquad n = 0, \pm 1, \pm 2, \ldots. \tag{15.78}$$

Thus, the radiation pattern has a single lobe for $L < \lambda_0$. For $L > \lambda_0$ the radiation pattern exhibits a primary lobe bounded by $n = \pm 1$ and secondary lobes on either side of the primary lobe. Determine the number of lobes for $L = 3\lambda_0$. Using the area under $g(\theta)$ in your plot for $L = 3\lambda_0$ qualitatively estimate the percentage of power radiated into the primary lobe.

2. (**20 points.**) The magnetic field associated to radiation fields is given by

$$c\mathbf{B}(\mathbf{r}, t) = -\hat{\mathbf{r}} \times \frac{\mu_0 c}{4\pi}\frac{1}{r}\int d^3r' \left\{\frac{1}{c}\frac{\partial}{\partial t'}\mathbf{J}(\mathbf{r}', t')\right\}_{t'=t_r}, \tag{15.79}$$

where the contribution to the field comes at the retarded time

$$t_r = t - \frac{r}{c} + \hat{\mathbf{r}} \cdot \frac{\mathbf{r}'}{c}. \tag{15.80}$$

The associated electric field is given by

$$\mathbf{E}(\mathbf{r}, t) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r}, t), \tag{15.81}$$

and satisfies

$$c\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, t). \tag{15.82}$$

From the flux of electromagnetic energy density (the Poynting vector) $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ we deduce the power dP radiated into the solid angle $d\Omega$ to be

$$dP = \lim_{r\to\infty} r^2 d\Omega\, \hat{\mathbf{r}} \cdot \mathbf{S}. \tag{15.83}$$

Using $\hat{\mathbf{r}} \cdot \mathbf{S} = \hat{\mathbf{r}} \cdot (\mathbf{E} \times \mathbf{H}) = (\hat{\mathbf{r}} \times \mathbf{E}) \cdot \mathbf{H}$ show that this leads to the expression

$$\frac{\partial P}{\partial \Omega} = \lim_{r\to\infty} \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\left|\frac{c\mathbf{B}(\mathbf{r}, t)}{\frac{\mu_0 c}{4\pi}\frac{1}{r}}\right|^2 = \lim_{r\to\infty} \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\left|\hat{\mathbf{r}} \times \boldsymbol{\iota}\right|^2, \tag{15.84}$$

where we defined the effective current (with direction), using the Greek letter iota,

$$\boldsymbol{\iota}\left(\hat{\mathbf{r}}, t - \frac{r}{c}\right) = \int d^3r' \left\{ \frac{1}{c} \frac{\partial}{\partial t'} \mathbf{J}(\mathbf{r}', t') \right\}_{t'=t_r}. \tag{15.85}$$

Verify that $c\mathbf{B}(\mathbf{r}, t) / \left(\frac{\mu_0 c}{4\pi} \frac{1}{r}\right)$ has the dimensions of current. Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\varepsilon_0}} = 29.9792458\,\Omega \tag{15.86}$$

has the dimensions of resistance.

(a) Consider an antenna configuration consisting of parallel current carrying wires of length L, separated by distance a, described in detail by

$$\begin{aligned} \mathbf{J}(\mathbf{r}', t') &= \hat{\mathbf{z}}\, I_0 \sin\omega_0 t'\, \delta\left(x' + \frac{a}{2}\right) \delta(y') \theta(-L < 2z' < L). \\ &\quad + \hat{\mathbf{z}}\, I_0 \sin\omega_0 t'\, \delta\left(x' - \frac{a}{2}\right) \delta(y') \theta(-L < 2z' < L). \end{aligned} \tag{15.87}$$

The function θ equals 1 when it's argument is a true statement, and zero otherwise. Show that

$$\begin{aligned} \boldsymbol{\iota}\left(\hat{\mathbf{r}}, t - \frac{r}{c}\right) &= \hat{\mathbf{z}}\, 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c}\sin\theta\cos\phi\right)\right) \frac{\sin\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta}, \\ &\quad + \hat{\mathbf{z}}\, 2I_0 \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c}\sin\theta\cos\phi\right)\right) \frac{\sin\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta}, \end{aligned} \tag{15.88}$$

where $\omega_0/c = 2\pi/\lambda_0$. Then, evaluate the expression for the magnetic field.

(b) Using Eq. (15.84) show that

$$\begin{aligned} \frac{\partial P}{\partial \Omega} = P_0 \frac{\sin^2\theta}{\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} & \left[\cos\left(\omega_0\left(t - \frac{r}{c} - \frac{a}{2c}\sin\theta\cos\phi\right)\right)\right. \\ & \left. + \cos\left(\omega_0\left(t - \frac{r}{c} + \frac{a}{2c}\sin\theta\cos\phi\right)\right)\right]^2, \end{aligned} \tag{15.89}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \tag{15.90}$$

Evaluate the average power $\bar{P}$ radiated into a solid angle using

$$\frac{\partial \bar{P}}{\partial \Omega} = \frac{1}{T_0} \int_0^{T_0} dt\, \frac{\partial P}{\partial \Omega}, \tag{15.91}$$

where $\omega_0 = 2\pi/T_0$. Show that

$$\frac{\partial \bar{P}}{\partial \Omega} = P_0 \frac{\sin^2\theta}{\pi} \frac{\sin^2\left(\pi \frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta} 2\cos^2\left(\pi \frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{15.92}$$

Hint: Use the integral

$$\frac{1}{T_0} \int_0^{T_0} dt\, \cos^2(\omega_0 t + \delta) = \frac{1}{2}. \tag{15.93}$$

(c) For the case $L \ll \lambda_0$, use the approximation

$$\frac{\sin\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos\theta} \sim \pi\frac{L}{\lambda_0} \tag{15.94}$$

to obtain

$$\frac{\partial\bar{P}}{\partial\Omega} = P_0\frac{\sin^2\theta}{\pi}\left(\pi\frac{L}{\lambda_0}\right)^2 2\cos^2\left(\pi\frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{15.95}$$

(d) For the case $\lambda_0 \ll L$, if we restrict our observation region to $\theta \sim \pi/2$, the system has the characteristics of a two dimensional system. To bring this characteristic out we integrate over θ,

$$\frac{\partial\bar{P}}{\partial\phi} = P_0\int_0^\pi \sin\theta d\theta\frac{\sin^2\theta}{\pi}\frac{\sin^2\left(\pi\frac{L}{\lambda_0}\cos\theta\right)}{\cos^2\theta}2\cos^2\left(\pi\frac{a}{\lambda_0}\sin\theta\cos\phi\right). \tag{15.96}$$

Substitute

$$z = \pi\frac{L}{\lambda_0}\cos\theta \tag{15.97}$$

such that

$$-\sin\theta d\theta = \frac{dz}{(\pi L/\lambda_0)} \tag{15.98}$$

and use the approximations

$$\sin\theta = \sqrt{1-\frac{z^2}{(\pi L/\lambda_0)^2}} \sim 1 \tag{15.99}$$

and

$$\pi\frac{L}{\lambda_0} \to \infty \tag{15.100}$$

to derive

$$\frac{\partial\bar{P}}{\partial\phi} = P_0\left(\pi\frac{L}{\lambda_0}\right)2\cos^2\left(\pi\frac{a}{\lambda_0}\cos\phi\right)\frac{1}{\pi}\int_{-\infty}^{\infty}dz\frac{\sin^2 z}{z^2}. \tag{15.101}$$

Use the integral

$$\int_0^\infty dz\frac{\sin^2 z}{z^2} = \int_0^\infty dz\frac{\sin z}{z} = \frac{\pi}{2}. \tag{15.102}$$

Thus, derive the expression for the average power radiated per angle $d\phi$,

$$\frac{\partial\bar{P}}{\partial\phi} = P_0\left(\pi\frac{L}{\lambda_0}\right)2\cos^2\left(\pi\frac{a}{\lambda_0}\cos\phi\right). \tag{15.103}$$

Compare this with the formula for double-slit interference pattern obtained using the Huygens-Fresnel principle for the classical wave propagation of light.

15.2 Radiation: frequency domain

1. (**20 points.**) The spectral distribution of power radiated into a solid angle $d\Omega = d\phi\sin\theta d\theta$ during Čerenkov radiation, when a particle of charge q moves with uniform speed v in a medium with index of refraction

$$n = n_\varepsilon n_\mu, \qquad n_\varepsilon = \sqrt{\frac{\varepsilon(\omega)}{\varepsilon_0}}, \qquad n_\mu = \sqrt{\frac{\mu(\omega)}{\mu_0}}, \tag{15.104}$$

is given by the expression

$$\frac{\partial^2 P}{\partial\omega\partial\Omega} = \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right) 2n_\mu^2 q^2 \omega \left(\frac{n^2 v^2}{c^2} - 1\right) \delta\left(1 - \frac{nv}{c}\cos\theta\right), \tag{15.105}$$

where ω is the frequency of light. Čerenkov light of a given frequency is emitted on a cone of half-angle θ_c. Determine the expression for θ_c. Show that for small θ_c,

$$\theta_c \sim \sqrt{2\left(1 - \frac{c}{nv}\right)}. \tag{15.106}$$

15.2.1 Loop antenna

1. (**80 points.**) The magnetic field associated to radiation fields, in the frequency domain, is given by

$$c\mathbf{B}(\mathbf{r},\omega) = -\hat{\mathbf{r}} \times \mathbf{F}(\theta,\phi;\omega)\,\frac{e^{ikr}}{r}, \tag{15.107}$$

where

$$\mathbf{F}(\theta,\phi;\omega) = \frac{\mu_0}{4\pi}(-i\omega)\mathbf{J}(\mathbf{k},\omega), \tag{15.108}$$

where we have used the notation

$$\mathbf{k} = \frac{\omega}{c}\hat{\mathbf{r}}. \tag{15.109}$$

for insight in the context of Fourier transformation. The associated electric field is given by

$$\mathbf{E}(\mathbf{r},\omega) = -\hat{\mathbf{r}} \times c\mathbf{B}(\mathbf{r},\omega), \tag{15.110}$$

and satisfies

$$c\mathbf{B}(\mathbf{r},\omega) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r},\omega). \tag{15.111}$$

The total energy E radiated into the solid angle $d\Omega$ per unit (positive, $0 \leq \omega < \infty$) frequency range $d\omega$ is given by

$$\frac{\partial}{\partial\omega}\frac{\partial E}{\partial\Omega} = \frac{1}{\pi}\frac{r^2}{c\mu_0}\Big|c\mathbf{B}(\mathbf{r},\omega)\Big|^2. \tag{15.112}$$

(a) Show that

$$\frac{\partial}{\partial\omega}\frac{\partial E}{\partial\Omega} = \frac{1}{4\pi}\left(\frac{\mu_0 c}{4\pi}\right)\frac{1}{\pi}\left|\frac{\omega}{c}\hat{\mathbf{r}} \times \mathbf{J}(\mathbf{r},\omega)\right|^2. \tag{15.113}$$

Verify that $\omega J/c$ has the dimensions of charge. (Caution: J here is the Fourier transform of current density.) Thus, conclude that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{15.114}$$

has the dimensions of resistance. Quantum phenomena in electromagnetism is characterized by the Planck's constant h and the associated fine-structure constant

$$\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}, \tag{15.115}$$

a dimensionless physical constant. Verify that

$$\frac{\mu_0 c}{4\pi} = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}} = \alpha\frac{\hbar}{e^2} = 29.9792458\,\Omega. \tag{15.116}$$

(b) A loop antenna consists of a circular infinitely thin conductor of radius a carrying a time-dependent current. Let the circular conductor be centered at the origin and placed on the x-y plane such that

$$\mathbf{J}(\mathbf{r}', t') = \hat{\boldsymbol{\phi}}' I_0 \sin\omega_0 t' \, \delta(\rho' - a)\delta(z'), \tag{15.117}$$

where $\rho' = \sqrt{x'^2 + y'^2}$ and $\hat{\boldsymbol{\phi}}' = -\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi'$. Evaluate the Fourier transform of the current density using

$$\mathbf{J}(\mathbf{k}, \omega) = \int d^3r' \int dt' e^{-i\mathbf{k}\cdot\mathbf{r}'} e^{i\omega t'} \mathbf{J}(\mathbf{r}', t') \tag{15.118}$$

and show that

$$\mathbf{J}(\mathbf{k}, \omega) = \hat{\boldsymbol{\phi}} \, 2\pi^2 a I_0 \, \delta(\omega - \omega_0) \, J_1(ka\sin\theta), \tag{15.119}$$

where $J_n(x)$ is the Bessel function of first kind.
Hint: You are expected to encounter the following integral

$$\int_0^{2\pi} d\phi' e^{-ika\sin\theta\cos(\phi-\phi')} \big[-\hat{\mathbf{x}}\sin\phi' + \hat{\mathbf{y}}\cos\phi' \big]. \tag{15.120}$$

Substitute $\phi' - \phi = \phi''$ to obtain

$$\hat{\boldsymbol{\phi}} \int_0^{2\pi} d\phi'' \cos\phi'' e^{-ika\sin\theta\cos\phi''} - \hat{\boldsymbol{\rho}} \int_0^{2\pi} d\phi'' \sin\phi'' e^{-ika\sin\theta\cos\phi''}. \tag{15.121}$$

Use the integrals

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \cos\phi' \, e^{-ix\cos\phi'} = (-i) J_1(x) \tag{15.122}$$

and

$$\int_0^{2\pi} \frac{d\phi'}{2\pi} \sin\phi' \, e^{-ix\cos\phi'} = 0. \tag{15.123}$$

We also dropped the delta-function contribution associated to $\delta(\omega + \omega_0)$, because $0 \leq \omega < \infty$.

(c) Show that

$$\frac{\partial}{\partial\omega}\frac{\partial P}{\partial\Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka\sin\theta) \, \delta(\omega - \omega_0), \tag{15.124}$$

where

$$P_0 = \left(\frac{\mu_0 c}{4\pi}\right) I_0^2. \tag{15.125}$$

Here we used the interpretation

$$\delta(\omega-\omega_0)\delta(\omega-\omega_0) = \delta(\omega-\omega_0) \int_{-\infty}^{\infty} dt \, e^{i(\omega-\omega_0)t} = \delta(\omega-\omega_0) \int_{-\infty}^{\infty} dt = \delta(\omega-\omega_0) T, \tag{15.126}$$

where T is the infinite time for which the system is evolving. We used E/T to be the power P.

(d) Integration with respect to frequency yields the power radiated per unit solid angle

$$\frac{\partial P}{\partial\Omega} = P_0 \pi^2 (ka)^2 J_1^2(ka\sin\theta). \tag{15.127}$$

Plot the angular distribution of radiated power for $ka = 0.5, 2, 3, 4, 6$. Note that

$$ka = \frac{\omega_0}{c} a = 2\pi \frac{a}{\lambda_0}, \tag{15.128}$$

where λ_0 is the wavelength associated with the angular frequency ω_0.

Chapter 16

Electromagnetic scattering

16.1 Terminology

Electromagnetic scattering can be broadly classified as elastic and inelastic. In an elastic scattering the obstacle does not absorb or dissipate energy. Rayleigh scattering, Thompson scattering, and Mie scattering are examples of elastic scattering. Thompson scattering is the regime when the energy of the incident wave is large relative to the characteristic energy (say the rest mass energy) of the obstacle, and thus is further classified as high energy scattering. In this spirit Rayleigh scattering is a low energy scattering, the energy of the incident wave is small in comparison to the characteristic energy of the obstacle. Energies in Mie scattering are intermediate between Rayleigh and Thompson scattering. In contrast, in an inelastic scattering the obstacles absorb or dissipate energy. Raman scattering is the inelastic version of low energy Rayleigh scattering, and Compton scattering is the inelastic version of the high energy Thompson scattering.

	Elastic	Inelastic
Low energy	Rayleigh	Raman
High energy	Thompson	Compton

Table 16.1: A simple classification of scattering processes

16.2 Green dyadic for Maxwell equations

Electromagnetic properties of materials are characterized by the following: (a) charge density and current density; (b) permanent electric and magnetic dipole moment density; and (c) induced electric and magnetic dipole moment density. This article will discuss neutral materials with no permanent electric and magnetic dipole moment densities. The induced electric and magnetic dipole moments are described by the electric suseptibility

$$\boldsymbol{\chi}(\mathbf{r},\omega) = \frac{\boldsymbol{\varepsilon}(\mathbf{r},\omega)}{\varepsilon_0} - \mathbf{1} \tag{16.1}$$

and the magnetic suseptibility

$$\boldsymbol{\chi}_m(\mathbf{r},\omega) = \frac{\boldsymbol{\mu}(\mathbf{r},\omega)}{\mu_0} - \mathbf{1}. \tag{16.2}$$

Here $\boldsymbol{\varepsilon}(\mathbf{r},\omega)$ is the electric permittivitty of the material and ε_0 is the electric permittivitty of vacuum, and the ratio $\boldsymbol{\varepsilon}(\mathbf{r},\omega)/\varepsilon_0$ is the dielectric constant of the material. Similarly, $\boldsymbol{\mu}(\mathbf{r},\omega)$ is the magnetic permeability of the material and μ_0 is the magnetic permeability of vacuum. For simplicity, we shall neglect magnetic permeability of the material in this discussion, which is not too limiting because magnetic effects are most often sub-dominant.

In the presence of an electric field, the dielectric constant of a material manifests as the induced polarization density $\mathbf{P}(\mathbf{r},t)$ of the material. It is a response to the applied electric field and is given by, assuming linear response,

$$\mathbf{P}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \boldsymbol{\chi}(\mathbf{r},t-t') \cdot \varepsilon_0 \mathbf{E}(\mathbf{r},t'). \tag{16.3}$$

The causal nature of this response is contained in the imposition $t > t'$, which is guarenteed when t is the upper limit of the integral in t'. The response contained in the integral relation of Eq. (16.3) is a convolution in time. This response when expressed in the Fourier frequency space is algebraic,

$$\mathbf{P}(\mathbf{r},\omega) = \boldsymbol{\chi}(\mathbf{r},\omega) \cdot \varepsilon_0 \mathbf{E}(\mathbf{r},\omega). \tag{16.4}$$

The simple form of the response in the frequency space comes at the price of the convoluted form in which causality is described in the frequency domain. That is, causality requires the susceptibility function to satisfy the Kramers-Kronig relations that connect the real and imaginary parts of the suscptibility function.

The constitutive fields are, assuming linear response,

$$\mathbf{D}(\mathbf{r},\omega) = \boldsymbol{\varepsilon}(\mathbf{r},\omega) \cdot \mathbf{E}(\mathbf{r},\omega) = \varepsilon_0 \mathbf{E}(\mathbf{r},\omega) + \mathbf{P}(\mathbf{r},\omega), \tag{16.5a}$$

$$\mathbf{B}(\mathbf{r},\omega) = \boldsymbol{\mu}(\mathbf{r},\omega) \cdot \mathbf{H}(\mathbf{r},\omega) = \mu_0 \mathbf{H}(\mathbf{r},\omega) + \mu_0 \mathbf{M}(\mathbf{r},\omega). \tag{16.5b}$$

As mentioned earlier, we assume $\mathbf{M} = 0$ in this discussion.

Electromagnetic interactions of electrically polarizable materials are governed by macroscopic Maxwell equations. Since the polarization as a response to the electric field is algebraic in the frequency domain, we define the Fourier tranform of the electric field

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \mathbf{E}(\mathbf{r},\omega), \tag{16.6}$$

with the inverse relation

$$\mathbf{E}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \mathbf{E}(\mathbf{r},t), \tag{16.7}$$

and similarly for the magnetic field $\mathbf{B}(\mathbf{r},t)$. In the Fourier transformed frequency domain the Maxwell equations, in the absence of charges and currents, and in the absence of permanent electric and magnetic polarizations, in SI units, are

$$\boldsymbol{\nabla} \times \mathbf{E}(\mathbf{r},\omega) = i\omega \mathbf{B}(\mathbf{r},\omega), \tag{16.8a}$$

$$\boldsymbol{\nabla} \times \mathbf{H}(\mathbf{r},\omega) = -i\omega \mathbf{D}(\mathbf{r},\omega). \tag{16.8b}$$

The constitutive fields $\mathbf{D}(\mathbf{r},\omega)$ and $\mathbf{B}(\mathbf{r},\omega)$ in Eqs. (16.5) are divergenceless in this case. This is implicit in Eqs. (16.8) and verified by taking divergence in these equations,

$$\boldsymbol{\nabla} \cdot \mathbf{D}(\mathbf{r},\omega) = 0, \tag{16.9a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B}(\mathbf{r},\omega) = 0. \tag{16.9b}$$

The Maxwell equations, in Eqs. (16.8), together, imply the dyadic differential equation for the electric field

$$\left[\frac{c^2}{\omega^2} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}) - \mathbf{1}\right] \cdot \varepsilon_0 \mathbf{E}(\mathbf{r},\omega) = \boldsymbol{\chi}(\mathbf{r},\omega) \cdot \varepsilon_0 \mathbf{E}(\mathbf{r},\omega). \tag{16.10}$$

The magnetic field is then given in terms of the electric field using Eq. (16.8a). The dyadic differential equation in Eq. (16.10) in principle governs all phenomena involving electromagnetic fields and electrically polarizable materials.

It is insightful to rewrite the operator in Eq. (16.10) as

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}) = -\nabla^2 \left(\mathbf{1} - \frac{\boldsymbol{\nabla}\boldsymbol{\nabla}}{\nabla^2}\right), \tag{16.11}$$

which brings out the fact that it is a projection operator.

The dyadic differential equation guides the construction of the dyadic differential equation

$$\left[\frac{c^2}{\omega^2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1})-\mathbf{1}\right]\cdot\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega)=\mathbf{1}\delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{16.12}$$

for the free Green dyadic $\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}')$. Interpretting the free Green dyadic to be the inverse of the associated dyadic differential equation we recognize the symbolic relation

$$\boldsymbol{\Gamma}_0^{-1}=\left[\frac{c^2}{\omega^2}\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{1})-\mathbf{1}\right]. \tag{16.13}$$

Thus, the free Green dyadic is the inverse of the associated differential operator. Using this in Eq. (16.10) we can write

$$\boldsymbol{\Gamma}_0^{-1}\cdot\mathbf{E}=\boldsymbol{\chi}\cdot\mathbf{E}. \tag{16.14}$$

The residual electric field in the absence of the material, $\mathbf{E}_{\text{in}}$, obtained when $\boldsymbol{\chi}=0$, satisfies

$$\boldsymbol{\Gamma}_0^{-1}\cdot\mathbf{E}_{\text{in}}=0. \tag{16.15}$$

Together, we have

$$\boldsymbol{\Gamma}_0^{-1}\cdot(\mathbf{E}-\mathbf{E}_{\text{in}})=\boldsymbol{\chi}\cdot\mathbf{E}, \tag{16.16}$$

which can be rewritten as

$$(\mathbf{E}-\mathbf{E}_{\text{in}})=\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\mathbf{E}. \tag{16.17}$$

Thus, we have the solution

$$\mathbf{E}=\mathbf{E}_{\text{in}}+\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\mathbf{E}. \tag{16.18}$$

Explicitly, this is a integro-differential equation,

$$\mathbf{E}(\mathbf{r},\omega)=\mathbf{E}_{\text{in}}(\mathbf{r},\omega)+\int d^3r'\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega)\cdot\boldsymbol{\chi}(\mathbf{r}',\omega)\cdot\mathbf{E}(\mathbf{r}',\omega), \tag{16.19}$$

where $\mathbf{E}_{\text{in}}$ is the electric field in the absence of the material. The dyadic differential equation in Eq. (16.10) and the integro-differential equation in Eq. (16.19) are both difficult to solve, and exact closed form solutions are typically possible only for configurations with planar and spherical symmetry.

In the weak approximation defined by $\int d^3r\Gamma_0\chi\sim V\Gamma_0\chi\ll 1$, where V is the volume of the material and Γ_0 and χ are characterized by the specific configuration, we can use perturbation theory. Using iteration we have the multiple scattering interpretation

$$\mathbf{E}=\mathbf{E}_{\text{in}}+\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\mathbf{E}_{\text{in}}+\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\mathbf{E}_{\text{in}}+\ldots. \tag{16.20}$$

In the weak approximation we have

$$\mathbf{E}=\mathbf{E}_{\text{in}}+\boldsymbol{\Gamma}_0\cdot\boldsymbol{\chi}\cdot\mathbf{E}_{\text{in}}, \tag{16.21}$$

obtained by neglecting the higher order terms. Thus, in the weak approximation we have the solution

$$\mathbf{E}(\mathbf{r},\omega)=\mathbf{E}_{\text{in}}(\mathbf{r},\omega)+\int d^3r'\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega)\cdot\boldsymbol{\chi}(\mathbf{r}',\omega)\cdot\mathbf{E}_{\text{in}}(\mathbf{r}',\omega). \tag{16.22}$$

16.3 Free Green dyadic

The electric field is thus given in terms of the free Green dyadic that satisfies Eq. (16.12). To find the solution for the free Green dyadic we begin by writing the dyadic differential equation in Eq. (16.12) in the form

$$\left[\boldsymbol{\nabla}\boldsymbol{\nabla}-\mathbf{1}\left(\nabla^2+\frac{\omega^2}{c^2}\right)\right]\cdot\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega)=\frac{\omega^2}{c^2}\mathbf{1}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{16.23}$$

Taking the divergence of the above equation we learn that

$$\boldsymbol{\nabla}\cdot\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = -\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{16.24}$$

Together, we have

$$-\left(\nabla^2+\frac{\omega^2}{c^2}\right)\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla}+\frac{\omega^2}{c^2}\mathbf{1}\right)\delta^{(3)}(\mathbf{r}-\mathbf{r}'). \tag{16.25}$$

This leads to the construction of the differential equation

$$-(\nabla^2+k^2)G_0(\mathbf{r},\mathbf{r}';\omega) = \delta^{(3)}(\mathbf{r}-\mathbf{r}') \tag{16.26}$$

for the Green function $G_0(\mathbf{r},\mathbf{r}';\omega)$, where

$$k=\frac{\omega}{c}. \tag{16.27}$$

In terms of the free Green function we can write

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla}+k^2\mathbf{1}\big]G_0(\mathbf{r},\mathbf{r}';\omega) \tag{16.28}$$

The free Green function has the general solution

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = A\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}+B\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{16.29}$$

with the constraint

$$A+B=1. \tag{16.30}$$

The free Green function in the time domain takes the form

$$G_0(\mathbf{r}-\mathbf{r}',t-t') = A\frac{\delta\left(t-t'-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}+B\frac{\delta\left(t-t'+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{16.31}$$

For $t>t'$ only one of the terms is causal. Thus, to respect causality we choose $B=0$. So, we have the solution

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{16.32}$$

Using Eq. (16.44) we have the free Green dyadic

$$\boldsymbol{\Gamma}_0(\mathbf{r}-\mathbf{r}';\omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla}+\frac{\omega^2}{c^2}\mathbf{1}\right)\frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{16.33}$$

After the gradient operators are evaluated we have

$$\boldsymbol{\Gamma}_0(\mathbf{r};\omega) = \frac{e^{ikr}}{4\pi r^3}\Big[-u(ikr)\mathbf{1}+v(ikr)\hat{\mathbf{r}}\hat{\mathbf{r}}\Big], \tag{16.34}$$

where

$$u(x) = 1-x+x^2, \tag{16.35a}$$
$$v(x) = 3-3x+x^2. \tag{16.35b}$$

The free Green dyadic in time domain has the form

$$\boldsymbol{\Gamma}_0(\mathbf{r},t) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla}-\mathbf{1}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\delta\left(t-t'-\frac{r}{c}\right)}{4\pi r}, \tag{16.36}$$

which has the form

$$\boldsymbol{\Gamma}_0(\mathbf{r},t) = \frac{1}{4\pi r^3}\left[-\mathbf{1}\,u\left(-\frac{r}{c}\frac{\partial}{\partial t}\right)+\hat{\mathbf{r}}\hat{\mathbf{r}}\,v\left(-\frac{r}{c}\frac{\partial}{\partial t}\right)\right]\delta\left(t-\frac{r}{c}\right). \tag{16.37}$$

Problems

1. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega)$ satisfies the dyadic differential equation

$$\frac{c^2}{\omega^2}\left[\boldsymbol{\nabla}\boldsymbol{\nabla} - \mathbf{1}\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\right] \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{16.38}$$

(a) Show that the divergence of the free Green dyadic is

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = -\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{16.39}$$

(b) Substitute the divergence in the dyadic differential equation and derive

$$-\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \left(\boldsymbol{\nabla}\boldsymbol{\nabla} + \frac{\omega^2}{c^2}\mathbf{1}\right)\delta^{(3)}(\mathbf{r} - \mathbf{r}'). \tag{16.40}$$

(c) Construct the differential equation

$$-(\nabla^2 + k^2)G_0(\mathbf{r}, \mathbf{r}'; \omega) = \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{16.41}$$

for the Green function $G_0(\mathbf{r}, \mathbf{r}'; \omega)$, where

$$k = \frac{\omega}{c}. \tag{16.42}$$

The free Green function has the (causal) solution

$$G_0(\mathbf{r} - \mathbf{r}'; \omega) = \frac{e^{i\frac{\omega}{c}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}. \tag{16.43}$$

Show that the free Green dyadic can be expressed in terms of the free Green function as

$$\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = \left[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\right]G_0(\mathbf{r}, \mathbf{r}'; \omega) \tag{16.44}$$

(d) The free Green dyadic is a function of $\mathbf{r} - \mathbf{r}'$. Thus, we can choose $\mathbf{r}'$ to be the origin without any loss of generality. Substituting $\mathbf{r} \to \mathbf{r} - \mathbf{r}'$ at any moment of the calculation returns the dependence in $\mathbf{r}'$. Evaluate the gradient operators and show that, for $\mathbf{r}' = 0$,

$$\boldsymbol{\Gamma}_0(\mathbf{r}; \omega) = \frac{e^{ikr}}{4\pi r^3}\left[-u(ikr)\mathbf{1} + v(ikr)\hat{\mathbf{r}}\hat{\mathbf{r}}\right], \tag{16.45}$$

where

$$u(x) = 1 - x + x^2, \tag{16.46a}$$
$$v(x) = 3 - 3x + x^2. \tag{16.46b}$$

16.4 Background fields and scatterred fields

A scattering process, and many other processes, involves an incoming (or a background) electromagnetic field that induces electric polarization in the material which then acts as a source for a scattered electromagnetic field. Thus, the fields can be constructed to have the decomposition

$$\mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_{\rm in}(\mathbf{r}, \omega) + \mathbf{E}_s(\mathbf{r}, \omega), \tag{16.47a}$$
$$\mathbf{B}(\mathbf{r}, \omega) = \mathbf{B}_{\rm in}(\mathbf{r}, \omega) + \mathbf{B}_s(\mathbf{r}, \omega). \tag{16.47b}$$

Clearly, we require the contributions from $\mathbf{E}_s$ and $\mathbf{B}_s$ to go to zero in the limit $\boldsymbol{\chi} \to 0$. The incident fields also satisfy the Maxwell equations independently,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}_{\text{in}}(\mathbf{r}, \omega) &= i\omega \mathbf{B}_{\text{in}}(\mathbf{r}, \omega), & (16.48\text{a}) \\ \boldsymbol{\nabla} \times \mathbf{H}_{\text{in}}(\mathbf{r}, \omega) &= -i\omega \mathbf{D}_{\text{in}}(\mathbf{r}, \omega), & (16.48\text{b}) \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_{\text{in}}(\mathbf{r}, \omega) &= \varepsilon_0 \mathbf{E}_{\text{in}}(\mathbf{r}, \omega), & (16.49\text{a}) \\ \mathbf{B}_{\text{in}}(\mathbf{r}, \omega) &= \mu_0 \mathbf{H}_{\text{in}}(\mathbf{r}, \omega), & (16.49\text{b}) \end{aligned}$$

and the fields $\mathbf{E}$ and $\mathbf{B}$ satisfy the Maxwell equations (16.8). In conjunction, Eqs. (16.8) and Eqs. (16.48) imply the following Maxwell equations for the scattered fields,

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E}_s(\mathbf{r}, \omega) &= i\omega \mathbf{B}_s(\mathbf{r}, \omega), & (16.50\text{a}) \\ \boldsymbol{\nabla} \times \mathbf{H}_s(\mathbf{r}, \omega) &= -i\omega \big[\mathbf{D}_s(\mathbf{r}, \omega) + \mathbf{P}_{\text{in}}(\mathbf{r}, \omega)\big], & (16.50\text{b}) \end{aligned}$$

where

$$\begin{aligned} \mathbf{D}_s(\mathbf{r}, \omega) &= \boldsymbol{\varepsilon}(\mathbf{r}, \omega) \cdot \mathbf{E}_s(\mathbf{r}, \omega), & (16.51\text{a}) \\ \mathbf{B}_s(\mathbf{r}, \omega) &= \boldsymbol{\mu}(\mathbf{r}, \omega) \mathbf{H}_s(\mathbf{r}, \omega), & (16.51\text{b}) \end{aligned}$$

and the induced polarization is given as a response to the incident electric field,

$$\mathbf{P}_{\text{in}}(\mathbf{r}, \omega) = \varepsilon_0 \boldsymbol{\chi}(\mathbf{r}, \omega) \cdot \mathbf{E}_{\text{in}}(\mathbf{r}, \omega). \tag{16.52}$$

The Maxwell equations (16.50) imply the divergenceless equations

$$\begin{aligned} \boldsymbol{\nabla} \cdot \big[\mathbf{D}_s(\mathbf{r}, \omega) + \mathbf{P}_{\text{in}}(\mathbf{r}, \omega)\big] &= 0, & (16.53\text{a}) \\ \boldsymbol{\nabla} \cdot \mathbf{B}_s(\mathbf{r}, \omega) &= 0. & (16.53\text{b}) \end{aligned}$$

The Maxwell equations, in Eqs. (16.50), together, imply the dyadic differential equation for the scattered electric field

$$\Big[\frac{\omega^2}{c^2} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}) - \mathbf{1} - \boldsymbol{\chi}(\mathbf{r}, \omega)\Big] \cdot \varepsilon_0 \mathbf{E}_s(\mathbf{r}, \omega) = \mathbf{P}_{\text{in}}(\mathbf{r}, \omega). \tag{16.54}$$

The magnetic field is then given in terms of the electric field using Eq. (16.50a). Introducing the dyadic differential equation

$$\Big[\frac{\omega^2}{c^2} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}) - \mathbf{1} - \boldsymbol{\chi}(\mathbf{r}, \omega)\Big] \cdot \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) = \mathbf{1} \delta^{(3)}(\mathbf{r} - \mathbf{r}') \tag{16.55}$$

for Green dyadic $\boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega)$ we have the solution

$$\varepsilon_0 \mathbf{E}_s(\mathbf{r}, \omega) = \int d^3 r' \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \mathbf{P}_{\text{in}}(\mathbf{r}', \omega), \tag{16.56}$$

that can also be written as

$$\mathbf{E}_s(\mathbf{r}, \omega) = \int d^3 r' \boldsymbol{\Gamma}(\mathbf{r}, \mathbf{r}'; \omega) \cdot \boldsymbol{\chi}(\mathbf{r}', \omega) \cdot \mathbf{E}_{\text{in}}(\mathbf{r}', \omega). \tag{16.57}$$

This is identical to the solution we found earlier when we recognize the relation

$$\boldsymbol{\Gamma} = (1 - \boldsymbol{\Gamma}_0 \cdot \boldsymbol{\chi})^{-1} \cdot \boldsymbol{\Gamma}_0. = \boldsymbol{\Gamma}_0 + \boldsymbol{\Gamma}_0 \cdot \boldsymbol{\chi} \cdot \boldsymbol{\Gamma}_0 + \dots. \tag{16.58}$$

16.5 Far-field approximation

The far-field approximation amounts to

$$r' \ll r \tag{16.59}$$

when the observation point $\mathbf{r}$ is very far relative to the source point $\mathbf{r}'$. This leads to the approximation

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} \sim r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{16.60}$$

Thus, in the far-field asymptotic limit we can replace

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{16.61}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{16.62}$$

In this form we see the structure of the spherical outgoing wave e^{ikr}/r emerging. Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla} \to i\mathbf{k}'. \tag{16.63}$$

Thus, in the far-field approximation we have

$$(\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}) \to (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})k^2 = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})k^2, \tag{16.64}$$

which projects vectors in the plane normal to the radial direction. The free Green dyadic in the far-field approximation takes the form

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = -\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{1})\frac{k^2}{4\pi}\frac{e^{ikr}}{r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}. \tag{16.65}$$

Problems

1. (**20 points.**) The free Green dyadic $\boldsymbol{\Gamma}_0$ can be expressed in terms of the free Green function G_0 as

$$\boldsymbol{\Gamma}_0(\mathbf{r},\mathbf{r}';\omega) = \big[\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}\big]G_0(\mathbf{r},\mathbf{r}';\omega), \tag{16.66}$$

where

$$G_0(\mathbf{r}-\mathbf{r}';\omega) = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}. \tag{16.67}$$

In the far-field approximation,

$$r' \ll r, \tag{16.68}$$

when the observation point $\mathbf{r}$ is very far relative to the source point $\mathbf{r}'$, show that

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2rr'} \sim r - \hat{\mathbf{r}}\cdot\mathbf{r}'. \tag{16.69}$$

Thus, in the far-field asymptotic limit show that

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \to \frac{e^{ikr}}{4\pi r}\, e^{-i\mathbf{k}'\cdot\mathbf{r}'}, \tag{16.70}$$

where we introduced the notation

$$\mathbf{k}' = k\,\hat{\mathbf{r}}. \tag{16.71}$$

Further, the far-field approximation allows the replacement

$$\boldsymbol{\nabla} \to i\mathbf{k}'. \tag{16.72}$$

Thus, in the far-field approximation show that

$$(\boldsymbol{\nabla}\boldsymbol{\nabla} + k^2\mathbf{1}) \to (\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})k^2 = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1})k^2, \tag{16.73}$$

which projects vectors in the plane normal to the radial direction. Thus, show that the free Green dyadic in the far-field approximation takes the form

$$\boldsymbol{\Gamma}_0(\mathbf{r}, \mathbf{r}'; \omega) = -\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1}) \frac{k^2}{4\pi} \frac{e^{ikr}}{r} e^{-i\mathbf{k}'\cdot\mathbf{r}'}. \tag{16.74}$$

16.6 Sommerfeld boundary condition for scattering

A particular phenomenon of interest is chosen, out of the multitude of processes governed by this dyadic equation, by specifying the boundary conditions satisfied by the fields. In this manner, an electromagnetic scattering process is characterized by the (Sommerfeld) boundary conditions

$$\lim_{r\to\infty} \mathbf{E}(\mathbf{r}, \omega) = \mathbf{E}_{\text{in}}(\mathbf{r}, \omega) + \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1}) \cdot \mathbf{K}(\theta, \phi, \omega) \frac{e^{ikr}}{r}, \tag{16.75a}$$

$$\lim_{r\to\infty} c\mathbf{B}(\mathbf{r}, \omega) = c\mathbf{B}_{\text{in}}(\mathbf{r}, \omega) - (\hat{\mathbf{r}} \times \mathbf{1}) \cdot \mathbf{K}(\theta, \phi, \omega) \frac{e^{ikr}}{r}, \tag{16.75b}$$

where the scattering amplitude, up to a scale, is

$$\mathbf{K}(\hat{\mathbf{r}}, \omega) \equiv \mathbf{K}(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \int d^3r'\, e^{-i\mathbf{k}'\cdot\mathbf{r}'} \boldsymbol{\chi}(\mathbf{r}', \omega) \cdot \mathbf{E}(\mathbf{r}', \omega). \tag{16.76}$$

These boundary conditions are dictated by the requirement that the total electric field $\mathbf{E}(\mathbf{r}, \omega)$ has a decomposition as in Eqs. (16.47), that involves an incident wave $\mathbf{E}_{\text{in}}(\mathbf{r}, \omega)$ and a scattered field $\mathbf{E}_s(\mathbf{r}, \omega)$. The incident wave is the part of the total field that is independent of the obstacle. In the far-field region it is deduced that the scattered field will have the characteristics of a spherical outgoing wave. It is further infered that the scattered electric and magnetic fields are tangent to the spherical dome of infinitely large radius enclosing the obstacle. This information is explicitly brought out in the boundary conditions of Eqs. (16.76) with the projection operators $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1})$ and $(\hat{\mathbf{r}} \times \mathbf{1})$. Recall, $\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{1}) = -(\mathbf{1} - \hat{\mathbf{r}}\hat{\mathbf{r}})$. The remaining structure of the scattered field that is completely dependent on the property of the obstacle is captured in the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ up to a scale dependent on the magnitude of incident wave. The scattering amplitude has the dimension of length, after we separate the dimension of the incident electric field (in frequency domain). For completeness, we repeat that the electric field in Eq. (16.76) is determined by solving the dyadic differential equation in Eq. (16.10),

$$\left[\frac{c^2}{\omega^2} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{1}) - \mathbf{1}\right] \cdot \mathbf{E}(\mathbf{r}, \omega) = \boldsymbol{\chi}(\mathbf{r}, \omega) \cdot \mathbf{E}(\mathbf{r}, \omega), \tag{16.77}$$

satisfying boundary conditions in Eq. (16.75). The magnetic field is then determined in terms of the electric field to be

$$c\mathbf{B}(\mathbf{r}, \omega) = \hat{\mathbf{r}} \times \mathbf{E}(\mathbf{r}, \omega). \tag{16.78}$$

In scattering problems, unlike bound state problems, we do not require $\mathbf{E}(\mathbf{r}, \omega)$ to go to zero in the far-field regions at $r \to \infty$. On the contrary, the quantity $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ containing the description of the scattered field in Eq. (16.75), which is the central quantity of interest in a scattering problem is completely contained in the asymptotic $r \to \infty$ part of $\mathbf{E}(\mathbf{r}, \omega)$. Observe that the far-field approximation is a means to impose the boundary condition for a scattering process, and thus is a defining property of the scattering problem. We recognize that the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ is evaluated in the far-field asymptotic region ($r \to \infty$) while the contributions to the scattering amplitude, as per Eq. (16.76), comes from short-range where $\boldsymbol{\chi}(\mathbf{r}, \omega)$ is nonzero, that is, from regions inside the obstacle. To summarize, we have formulated the scattering problem in terms of $\mathbf{K}(\hat{\mathbf{r}}, \omega)$ given by Eq. (16.76) in which the electric field is determined by solving the dyadic integral equation in Eq. (16.77) satisfying boundary conditions in Eq. (16.75).

16.7 Scattering amplitude

Let the incident wave be a monochromatic plane wave of frequency ω with fields in time domain given by

$$\mathbf{E}_{\text{in}}(\mathbf{r},t) = \text{Re}\,\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{16.79a}$$

$$\mathbf{B}_{\text{in}}(\mathbf{r},t) = \text{Re}\,\mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{16.79b}$$

These incident fields satisfy Maxwell equations, independently, which implies $k = \frac{\omega}{c}$, and the direction of $\mathbf{k}$ is constrained by with $\mathbf{k}\cdot\mathbf{E}_0 = 0$ and $\mathbf{k}\cdot\mathbf{B}_0 = 0$. In the frequency domain the incident field is given by

$$\mathbf{E}_0(\mathbf{r},\omega) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{16.80a}$$

$$\mathbf{B}_0(\mathbf{r},\omega) = \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{16.80b}$$

In the weak approximation we can replace the incident electric field for the total electric field in the expression for $\mathbf{K}(\hat{\mathbf{r}},\omega)$ given by Eq. (16.76)

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = -\frac{k^2}{4\pi}\int d^3r'\, e^{-i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}'}\boldsymbol{\chi}(\mathbf{r}',\omega)\cdot\mathbf{E}_0. \tag{16.81}$$

Recognizing the integral to be the Fourier transform in the spatial domain we can write

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = -\frac{k^2}{4\pi}\boldsymbol{\chi}(\mathbf{k}'-\mathbf{k},\omega)\cdot\mathbf{E}_0. \tag{16.82}$$

For the case of isotropic scatterer we have

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = \mathbf{E}_0\, f(\hat{\mathbf{r}},\omega), \tag{16.83}$$

in terms of the scattering amplitude

$$f(\theta,\phi,\omega) = f(\hat{\mathbf{r}},\omega) = -\frac{k^2}{4\pi}\chi(\mathbf{k}'-\mathbf{k},\omega). \tag{16.84}$$

It can be verified that the scattering amplitude has the dimensions of length. Note that electric susceptibility $\boldsymbol{\chi}(\mathbf{r}',\omega)$ is dimensionless, while its Fourier transform $\boldsymbol{\chi}(\mathbf{q},\omega)$ has the dimensions of volume.

16.8 Scattering cross section

The statement of conservation of electromagnetic energy in the time domain is

$$\frac{\partial}{\partial t}U(\mathbf{r},t) + \boldsymbol{\nabla}\cdot\mathbf{S}(\mathbf{r},t) + \mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = 0, \tag{16.85}$$

where $U(\mathbf{r},t)$ is the electromagnetic energy density introduced by Sommerfeld and Brillouin and $\mathbf{S}(\mathbf{r},t)$ is the flux of electromagnetic energy density or the Poynting vector given by

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t). \tag{16.86}$$

Let us define the time average of the rate of change of electromagnetic energy density at a point as the average power density

$$p(\mathbf{r}) = \lim_{\tau\to\infty}\frac{1}{\tau}\int_{-T}^{T} dt\frac{\partial}{\partial t}U(\mathbf{r},t) = \lim_{\tau\to\infty}\frac{U(\mathbf{r},T)-U(\mathbf{r},-T)}{\tau}, \tag{16.87}$$

where $\tau = 2T$ is the (infinite) time for which the system evolves. Thus, we have

$$p(\mathbf{r}) + \frac{1}{\tau}\int_{-\infty}^{\infty} dt\boldsymbol{\nabla}\cdot\mathbf{S}(\mathbf{r},t) + p_{\text{abs.}}(\mathbf{r}) = 0, \tag{16.88}$$

where

$$p_{\text{ch.}}(\mathbf{r}) = \frac{1}{\tau}\int_{-\infty}^{\infty} dt\,\mathbf{j}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) = 0. \tag{16.89}$$

Integrating over all space and using divergence theorem we have

$$P + \frac{1}{\tau}\int_{-\infty}^{\infty} dt \oint d\Omega r^2 \hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},t) + P_{\text{abs.}} = 0. \tag{16.90}$$

The Poynting vector is a bilinear construction in terms of the fields. Thus, using Plancherel theorem we have

$$\int_{-\infty}^{\infty} dt\,\mathbf{S}(\mathbf{r},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathbf{S}(\mathbf{r},\omega), \tag{16.91}$$

where

$$\mathbf{S}(\mathbf{r},\omega) = \frac{1}{2}\Big[\mathbf{E}(\mathbf{r},\omega)^*\times\mathbf{H}(\mathbf{r},\omega) + \mathbf{E}(\mathbf{r},\omega)\times\mathbf{H}(\mathbf{r},\omega)^*\Big], \tag{16.92}$$

where we introduced symmetrization under complex conjugation. This symmetrization is necessary whenever these construction appear outside the frequency integral. This symmetrization will be not written explicitly from now onwards to avoid clutter in the expressions. The total flux of electromagnetic energy, obtained by integrating over all time, in conjunction with the Plancherel theorem provides the frequency distribution of the total flux in $\mathbf{S}(\mathbf{r},\omega)$. Thus, the statement of conservation of energy in the frequency domain dictates the frequency distribution of the power to be

$$\frac{\partial P}{\partial\omega} + \frac{1}{2\pi\tau}\oint d\Omega r^2\hat{\mathbf{r}}\cdot\mathbf{S}(\mathbf{r},\omega) + \frac{\partial P_{\text{abs.}}}{\partial\omega} = 0. \tag{16.93}$$

The decomposition in Eqs. (16.47) associated with a scattering process introduces the following decomposition in the frequency distribution of the total flux to have the form

$$\mathbf{S}(\mathbf{r},\omega) = \mathbf{S}_{\text{in}}(\mathbf{r},\omega) + \mathbf{S}_s(\mathbf{r},\omega) + \mathbf{S}_{\text{damp.}}(\mathbf{r},\omega), \tag{16.94}$$

where

$$\begin{aligned}
\mathbf{S}_{\text{in}}(\mathbf{r},\omega) &= \mathbf{E}_{\text{in}}(\mathbf{r},\omega)^*\times\mathbf{H}_{\text{in}}(\mathbf{r},\omega), && (16.95\text{a})\\
\mathbf{S}_s(\mathbf{r},\omega) &= \mathbf{E}_s(\mathbf{r},\omega)^*\times\mathbf{H}_s(\mathbf{r},\omega), && (16.95\text{b})\\
\mathbf{S}_{\text{damp.}}(\mathbf{r},\omega) &= \mathbf{E}_{\text{in}}(\mathbf{r},\omega)^*\times\mathbf{H}_s(\mathbf{r},\omega) + \mathbf{E}_s(\mathbf{r},\omega)^*\times\mathbf{H}_{\text{in}}(\mathbf{r},\omega), && (16.95\text{c})
\end{aligned}$$

with symmetrization under complex conjugation understood implicitly. Let us define the following cross sections

$$\begin{aligned}
\sigma_{\text{in}} &= \frac{1}{|\mathbf{S}_{\text{in}}|}\oint d\Omega\, r^2\hat{\mathbf{r}}\cdot\mathbf{S}_{\text{in}}(\mathbf{r},\omega), && (16.96\text{a})\\
\sigma_{\text{scatt.}} &= \frac{1}{|\mathbf{S}_{\text{in}}|}\oint d\Omega\, r^2\hat{\mathbf{r}}\cdot\mathbf{S}_s(\mathbf{r},\omega), && (16.96\text{b})\\
\sigma_{\text{damp.}} &= \frac{1}{|\mathbf{S}_{\text{in}}|}\oint d\Omega\, r^2\hat{\mathbf{r}}\cdot\mathbf{S}_{\text{damp.}}(\mathbf{r},\omega), && (16.96\text{c})
\end{aligned}$$

and the total cross section

$$\sigma_{\text{tot.}} = \sigma_{\text{in}} + \sigma_{\text{scatt.}} + \sigma_{\text{damp.}} \tag{16.97}$$

such that the power spectrum is given by

$$\frac{\partial P}{\partial\omega} + \frac{|\mathbf{S}_{\text{in}}|}{2\pi\tau}\sigma_{\text{tot.}} + \frac{\partial P_{\text{abs.}}}{\partial\omega} = 0. \tag{16.98}$$

Here

$$|\mathbf{S}_{\text{in}}| = \frac{|E_0|^2}{\mu_0 c} \tag{16.99}$$

has the dimensions of energy per unit area times time.

In general we have

$$\sigma_{\text{in}} = 0 \tag{16.100}$$

because it involves an angular integration of $\hat{\mathbf{r}} \cdot \mathbf{S}_{\text{in}} \sim \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \cos\theta$. In general we have

$$\sigma_{\text{scatt.}} = \oint d\Omega\, r^2 \frac{|c\mathbf{B}_s(\mathbf{r},\omega)|^2}{|cB_0|^2}. \tag{16.101}$$

In terms of the scattering amplitude $\mathbf{K}(\hat{\mathbf{r}},\omega)$ we have the scattering cross section

$$\sigma_{\text{scatt.}} = \oint d\Omega\, \frac{|\hat{\mathbf{r}} \times \mathbf{K}(\hat{\mathbf{r}},\omega)|^2}{|E_0|^2} = \oint d\Omega\, \frac{\left[|\mathbf{K}|^2 - |\hat{\mathbf{r}} \cdot \mathbf{K}|^2\right]}{|E_0|^2}. \tag{16.102}$$

The energy content of the scattered radiation is supplied by the energy lost by the incident beam in the dielectric material. This energy lost is given by

$$\sigma_{\text{damp.}} = -\frac{4\pi}{k} \frac{\text{Im}\left[\mathbf{E}_0^* \cdot \mathbf{K}(\hat{\mathbf{z}},\omega)\right]}{|E_0|^2}. \tag{16.103}$$

This term is interpreted as dissipation of energy inside the dielectric medium due to the impediment of the incident beam. The cause of this dissipation is radiation damping, or reaction force experienced by the induced dipoles due to radiation. Together, we have

$$\sigma_{\text{tot.}} = \sigma_{\text{scatt.}} + \sigma_{\text{damp.}} \tag{16.104}$$

which is the optical theorem. In summary,

$$\frac{\partial P}{\partial \omega} + \frac{|\mathbf{S}_{\text{in}}|}{2\pi\tau}\left[\sigma_{\text{scatt.}} + \sigma_{\text{damp.}} + \sigma_{\text{abs.}}\right] = 0. \tag{16.105}$$

For an isotropic scatterer we have the cross section

$$\sigma_{\text{scatt.}} = \oint d\Omega \left[1 - (\hat{\mathbf{r}} \cdot \hat{\mathbf{E}}_0)^2\right] |f(\theta,\phi,\omega)|^2. \tag{16.106}$$

In addition if the material has azimuthal symmetry we obtain

$$\sigma_{\text{scatt.}} = \int_0^\pi \sin\theta d\theta\, (1 + \cos^2\theta) |f(\theta,\omega)|^2. \tag{16.107}$$

16.8.1 Scattering off a point polarizable atom

A point polarizable atom is described by the susceptibility

$$\boldsymbol{\chi}(\mathbf{r},\omega) = 4\pi \boldsymbol{\alpha}(\omega)\, \delta^{(3)}(\mathbf{r} - \mathbf{s}), \tag{16.108}$$

where $\mathbf{s}$ is the position of the obstacle, that has been, for convenience, modeled as a point obstacle using δ-functions. Here $\boldsymbol{\alpha}$ is the polarizability of the obstacle, and has dimensions of length-cube. Show that these replacements lead to the scattering amplitude

$$\mathbf{K}(\hat{\mathbf{r}},\omega) = -k^2 \boldsymbol{\alpha}(\omega) \cdot \mathbf{E}_0\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{s}}. \tag{16.109}$$

For obstacles with isotropic polarizabilities we have $\boldsymbol{\alpha}(\omega) = \mathbf{1}\alpha(\omega)$ and the scattering amplitude takes the form

$$f(\hat{\mathbf{r}},\omega) = -k^2\, \alpha(\omega)\, e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{s}}. \tag{16.110}$$

The scattering cross section is, in the absence of absorption,

$$\sigma_{\text{scatt.}} = 4\pi k\, \text{Im}\, \alpha(\omega). \tag{16.111}$$

16.8.2 Scattering off a thin film

If the obstacles are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$,

$$\boldsymbol{\chi}(\mathbf{r}, \omega) = 4\pi \boldsymbol{\lambda}(\mathbf{s})\, \delta(z), \tag{16.112}$$

where the δ-function has been used to describe the assumption that the obstacles in a thin film are confined to a plane, $z = 0$ here. Once the obstacles are restricted to be on a plane, we can choose the direction of incidence $\mathbf{k}$ of the plane wave to be normal to the plane. That is, $\mathbf{k} \cdot \mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point obstacles on the plane. Further, notice that in this special case the electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations the scattering amplitude is given by, for isotropic polarizabilities,

$$f(\hat{\mathbf{r}}, \omega) = -k^2 \int d^2s\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}} \lambda(\mathbf{s}). \tag{16.113}$$

For a disc of radius R centered at position $\mathbf{s}_0$ with uniform polarizability per unit area λ we can complete the integrals to obtain

$$f(\hat{\mathbf{r}}, \omega) = -\lambda k^2 \pi R^2 2 \frac{J_1(kR\sin\theta)}{kR\sin\theta}\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \tag{16.114}$$

Here we used the integral representation of zeroth order Bessel function of the first kind

$$J_0(t) = \int_0^{2\pi} \frac{d\phi}{2\pi}\, e^{it\cos\phi} \tag{16.115}$$

and the identity

$$\int_0^b t dt J_0(t) = bJ_1(b). \tag{16.116}$$

Note the limiting value

$$\lim_{x\to 0} \frac{J_1(x)}{x} = \frac{1}{2}, \tag{16.117}$$

which guarantees a well defined value for the scattering amplitude at $\theta = 0$. We observe the interesting feature that the scattering amplitude at $\theta = 0$ is entirely given by the area of the disc.

For a ring of radius R centered at position $\mathbf{s}_0$ with uniform polarizability per unit length σ we can similarly complete the integrals to obtain

$$f(\theta, \phi, \omega) = -\sigma k^2 2\pi R J_0(kR\sin\theta) e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \tag{16.118}$$

For multiple circles the scattering amplitude obeys linear superposition in the weak approximation. Thus,

$$f(\theta, \phi, \omega) = -\sigma k^2 \sum_i 2\pi R_i J_0(kR_i\sin\theta) e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_i}, \tag{16.119}$$

where i represents the sum over multiple circles of radii R_i centered at positions $\mathbf{s}_i$, which are all assumed to have identical uniform isotropic polarizability per unit length σ. This sets the stage for scattering off Ford circles.

Problems

1. (**20 points.**) The scattering amplitude off a scatterer of susceptibility $\chi(\mathbf{r}, \omega)$ is given by

$$f(\theta, \phi, \omega) = -\frac{k^2}{4\pi} \chi(\mathbf{k}' - \mathbf{k}, \omega), \tag{16.120}$$

where $\chi(\mathbf{q}, \omega)$ is the Fourier transform of $\chi(\mathbf{r}, \omega)$,

$$\chi(\mathbf{q}, \omega) = \int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}} \chi(\mathbf{r}, \omega). \tag{16.121}$$

If the scatterers are confined on a plane, say $z = 0$, then it is convenient to define polarizability per unit area $\boldsymbol{\lambda} = \boldsymbol{\alpha}/\text{Area}$, and the associated susceptibility

$$\boldsymbol{\chi}(\mathbf{r}, \omega) = 4\pi \boldsymbol{\lambda}(\mathbf{s})\, \delta(z), \tag{16.122}$$

where the δ-function has been used to describe the assumption that the obstacles in a thin film are confined to the $z = 0$ plane here. Once the scatterers are restricted on a plane, we can choose the direction of incidence of the plane wave $\mathbf{k}$ to be normal to the plane constituting the scatterers. That is, $\mathbf{k} \cdot \mathbf{s} = 0$, where $\mathbf{s}$ are the positions of the point scatterers on the plane $z = 0$. Also, note that the amplitude of the incident electric field $\mathbf{E}_0$ is independent of the position $\mathbf{s}$. Using these considerations show that the scattering amplitude, for isotropic polarizabilities, $\boldsymbol{\lambda}(\mathbf{s}) = \mathbf{1}\lambda(\mathbf{s})$, is given by

$$f(\theta, \phi, \omega) = -k^2 \int d^2 s\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}} \lambda(\mathbf{s}). \tag{16.123}$$

For a disc of radius R centered at position $\mathbf{s}_0$ with uniform polarizability per unit area λ complete the integrals to obtain

$$f(\theta, \phi, \omega) = -\lambda k^2 \pi R^2 2 \frac{J_1(kR\sin\theta)}{kR\sin\theta}\, e^{ik\hat{\mathbf{r}}\cdot\mathbf{s}_0}. \tag{16.124}$$

16.9 Scattering off Ford circles

Ford circles consists of the set of circles of radii

$$R_i = \frac{R}{2n^2} \tag{16.125}$$

with center at the respective positions

$$\mathbf{s}_i = \left(\frac{m}{n} R, \frac{R}{2n^2} \right), \tag{16.126}$$

where the i-th circle is labeled using two integers n and m. Here n takes on values from 1 to ∞ and m takes values from 1 to n, with the requirement that the greatest common divisor of n and m should be 1, that is n and m and coprime. To this set we add one more circle at position $(0, R/2)$ of radius $R/2$, which could be associated to $n = 0$ with no associated value for m.

The sum over all the Ford circles is characterized by

$$\sum_i = {\sum_{n=0}^{\infty}}' \sum_{m=1}^{n} \delta_{1,\gcd(n,m)}, \tag{16.127}$$

where the Kronecker delta symbol δ_{ij} is non-zero only when $i = j$, and is unity when it is non-zero. Here the prime on the summation over n reminds us that the $n = 0$ contribution should be treated in a special manner, in the sense that it has no associated sum over m.

The scattering amplitude for scattering off ford circles is given by

$$\begin{aligned} f(\theta, \phi, \omega) \;=\; -\sigma k^2 2\pi \Bigg[& \frac{R}{2} J_0\left(k\frac{R}{2}\sin\theta \right) e^{i\frac{1}{2}kR\sin\theta\sin\phi} \\ & + \sum_{n=1}^{\infty} \frac{R}{2n^2} J_0\left(k\frac{R}{2n^2}\sin\theta \right) e^{ik\frac{R}{2n^2}\sin\theta\sin\phi} \sum_{m=1}^{n} \delta_{1,\gcd(n,m)} e^{ikR\sin\theta\cos\phi\frac{m}{n}} \Bigg]. \end{aligned} \tag{16.128}$$

For $\phi = \frac{\pi}{2}$ the exponential term containing n contributes unity, and for this case the sum over n is, by definition, the Euler's totient function

$$\varphi(n) = \sum_{m=1}^{n} \delta_{1,\gcd(n,m)}. \tag{16.129}$$

Thus, we have

$$f(\theta, \tfrac{\pi}{2}, \omega) = -\sigma k^2 (2\pi R) \frac{1}{2} \left[J_0 \left(k \frac{R}{2} \sin\theta \right) e^{i \frac{1}{2} kR \sin\theta} + \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} J_0 \left(k \frac{R}{2n^2} \sin\theta \right) e^{ik \frac{R}{2n^2} \sin\theta} \right]. \tag{16.130}$$

This calls for the definition of the perimeter function

$$P(t) = \frac{1}{2} \left[J_0(t) e^{it} + \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} J_0 \left(\frac{t}{n^2} \right) e^{i \frac{t}{n^2}} \right] \tag{16.131}$$

in terms of which we can write the scattering amplitude as

$$f(\theta, \tfrac{\pi}{2}, \omega) = -\sigma k^2 (2\pi R) \, P \left(\frac{1}{2} kR \sin\theta \right). \tag{16.132}$$

The perimeter function can be evaluated in closed form for $t = 0$ corresponding to $\theta = 0$,

$$P(0) = \frac{1}{2} \left[1 + \frac{\zeta(1)}{\zeta(2)} \right]. \tag{16.133}$$

In an attempt to evaluate $P(t)$ in general we consider the sum

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} J_0 \left(\frac{t}{n^2} \right) e^{i \frac{t}{n^2}} &= \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i \frac{2t}{n^2} \sin^2 \frac{\phi}{2}} \\ &= \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{m=0}^{\infty} \frac{1}{m!} \left(i \frac{2t}{n^2} \sin^2 \frac{\phi}{2} \right)^m \\ &= \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (i2t)^m \frac{\Gamma(m + \frac{1}{2})}{[\Gamma(m+1)]^2} \frac{\zeta(2m+1)}{\zeta(2m+2)}. \end{aligned} \tag{16.134}$$

Similarly we can evaluate

$$J_0(t) e^{it} = \frac{1}{\sqrt{\pi}} \sum_{m=0}^{\infty} (i2t)^m \frac{\Gamma(m + \frac{1}{2})}{[\Gamma(m+1)]^2}. \tag{16.135}$$

Together, we have the series for the perimeter function

$$P(t) = \frac{1}{2\sqrt{\pi}} \sum_{m=0}^{\infty} (i2t)^m \frac{\Gamma(m + \frac{1}{2})}{[\Gamma(m+1)]^2} \left[1 + \frac{\zeta(2m+1)}{\zeta(2m+2)} \right]. \tag{16.136}$$

Chapter 17

Action for Maxwell fields

17.1 Action

1. (**20 points.**) (Refer Schwinger et al. problem 10.11.)
   In covariant notation, the action for the electromagnetic field interacting with a prescribed current $j^\mu = (c\rho, \mathbf{j})$ is
   $$W = \int d^4x \left[\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\mu_0} F^{\mu\nu} (\partial_\mu A_\nu - \partial_\nu A_\mu) + j^\mu A_\mu \right]. \tag{17.1}$$
   In the action the vector potential A_μ and the field strength tensor $F_{\mu\nu}$ are regarded as independent variables.
   (a) Derive
   $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{17.2}$$
   and
   $$\partial_\nu F^{\mu\nu} = \mu_0 \, j^\mu \tag{17.3}$$
   by requiring that W be stationary under independent variations in $F_{\mu\nu}$ and A_μ respectively.
      i. Further, derive the statement of conservation of charge,
      $$\partial_\mu j^\mu = 0. \tag{17.4}$$
      ii. By adding the null term
      $$\int d^4x \, \lambda \, \partial_\mu j^\mu \tag{17.5}$$
      to the action show that the action is invariant under the gauge transformation
      $$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{17.6}$$
      where λ is an arbitrary function of spacetime.
   (b) Consider a general coordinate transformation
   $$\bar{x}^\mu = x^\mu - \delta x^\mu. \tag{17.7}$$
   A scalar field $\phi(x)$ changes under such a transformation as
   $$\delta\phi(x) = \phi(x - \delta x) - \phi(x) = -\delta x^\lambda \partial_\lambda \phi. \tag{17.8}$$
   Because the action is invariant under a gauge transformation, we conclude that the vector potential A_μ responds to a general coordinate transformation as the derivative of a scalar field. Thus derive,
   $$\delta A_\mu = -(\partial_\mu \delta x^\lambda) A_\lambda - \delta x^\lambda \partial_\lambda A_\mu. \tag{17.9}$$

Further, derive the response of the field strength tensor $F_{\mu\nu}$ to a general coordinate transformation as

$$\delta F_{\mu\nu} = -(\partial_\mu \delta x^\lambda) F_{\lambda\nu} - (\partial_\nu \delta x^\lambda) F_{\mu\lambda} - \delta x^\lambda \partial_\lambda F_{\mu\nu}. \tag{17.10}$$

(c) Now consider a source-free region, where $j^\mu = 0$, and the fields vanish outside the space-time region in question. Assume now that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, so that

$$W = -\int d^4x \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{17.11}$$

Show that

$$\delta W = \int d^4x (\partial_\mu \delta x_\nu) t^{\mu\nu}, \tag{17.12}$$

where

$$t^{\mu\nu} = \frac{1}{\mu_0} F^{\mu\lambda} F^\nu{}_\lambda + g^{\mu\nu} \mathcal{L}. \tag{17.13}$$

(d) For $\delta x^\lambda =$ constant show that $\delta W = 0$.

(e) Use the action principle to show that $t^{\mu\nu}$ is conserved,

$$\partial_\mu t^{\mu\nu} = 0. \tag{17.14}$$

(f) Verify that t^{00} is the energy density,

$$t^{00} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 = U, \tag{17.15}$$

t^{i0} is the energy flux vector,

$$t^{i0} = \frac{1}{c}\mathbf{E} \times \mathbf{H} = \frac{1}{c}\mathbf{S}, \tag{17.16}$$

t^{0i} is the momentum density,

$$t^{0i} = c\mathbf{D} \times \mathbf{B} = c\mathbf{G}, \tag{17.17}$$

and t^{ij} is the momentum flux tensor,

$$\mathbf{t} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{17.18}$$

Thus, we have the conservation of energy

$$\frac{\partial U}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = 0 \tag{17.19}$$

and conservation of momentum

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{t} = 0. \tag{17.20}$$

(g) What is the trace of $t^{\mu\nu}$? What is the significance of that result?