Electromagnetism ④

I Differential vector calculus

① Gradient

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= d\vec{x} \cdot \vec{\nabla} f$$

Interpret magnitude and direction of $\vec{\nabla} f(x,y,z)$

② Divergence

$$\vec{\nabla} \cdot \vec{A}(x,y,z) = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$$

③ Curl

$$\vec{\nabla} \times \vec{A}(x,y,z) = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial}{\partial z} A_x - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$

II Integral vector calculus

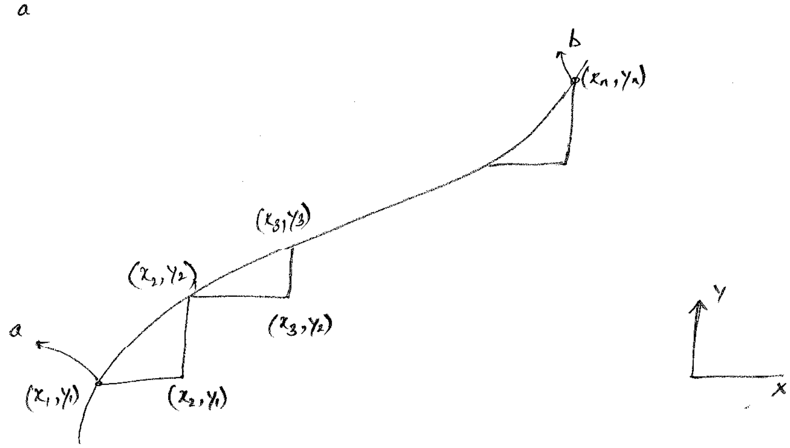
1. Line integral $\rightarrow$ $\int_{a\,\text{path}}^{b} \vec{A} \cdot d\vec{l}$

Surface integral $\rightarrow$ $\int_S d\vec{a} \cdot \vec{A}$

Volume integral $\rightarrow$ $\int_V d^3r \, \vec{A}$

2. Gradient theorem

$$\int_a^b d\vec{l} \cdot \vec{\nabla} f = f(b) - f(a)$$



$\rightarrow$ 2-D analog.

$f = f(x, y)$

③ $\int_a^b d\vec{l} \cdot \vec{\nabla} f = \int_a^b dx \frac{\partial f}{\partial x} + dy \frac{\partial f}{\partial y}$

$= dx \frac{f(x_2,y_1) - f(x_1,y_1)}{dx} + dy \frac{f(x_2,y_2) - f(x_2,y_1)}{dy}$

$+ \dots + \text{higher orders.}$

$= [f(x_2,y_2) - f(x_1,y_1)] + [f(x_3,y_3) - f(x_2,y_2)] + \dots$

$= f(x_n,y_n) - f(x_1,y_1)$

$= f(b) - f(a)$

④ $\int_a^b d\vec{l} \cdot \vec{\nabla} f = f(b) - f(a)$ is path independent.

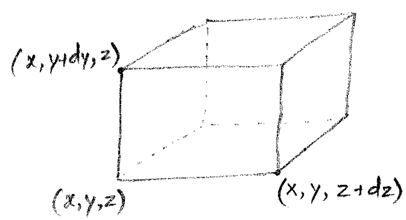
⑤ Try to prove:

$\left\{ \begin{matrix} \text{path independence of} \\ \int_a^b d\vec{l} \cdot \vec{A} \end{matrix} \right\} \Rightarrow \vec{\nabla} \times \vec{A} = 0$

(6) Divergence theorem:

$$\int_V d^3r \; \vec{\nabla}\cdot\vec{A} = \oint_S d\vec{a}\cdot\vec{A}$$

(7) Consider an infinitismal cube



(8) $$dx\,dy\,dz\left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right]$$

$$= dy\,dz\left[A_x(x+dx, y, z) - A_x(x,y,z)\right]$$
$$+ dx\,dz\left[A_y(x, y+dy, z) - A_y(x,y,z)\right]$$
$$+ dx\,dy\left[A_z(x,y,z+dz) - A_z(x,y,z)\right]$$

(9) Define $d\vec{a}$ (outgoing vector) and thus conclude

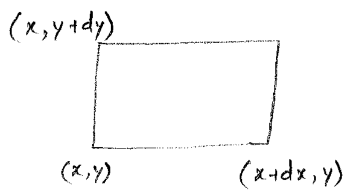
$$dx\,dy\,dz\;\vec{\nabla}\cdot\vec{A} = d\vec{a}\cdot\vec{A}$$

$$\Rightarrow \int_V d^3r\,(\vec{\nabla}\cdot\vec{A}) = \oint_S d\vec{a}\cdot\vec{A}$$ ($\because$ internal surfaces cancel out.)

(10) <u>Curl theorem</u> :

$$\int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = \oint d\vec{l} \cdot \vec{A}$$

(11) Consider an infinitismal square for simplification



(12) $$dx\,dy \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right]$$
$$= dy \left[A_y(x+dx,y) - A_y(x,y) \right] + dx \left[A_x(x,y+dy) - A_x(x,y) \right]$$

(13) Define $d\vec{l}$ (anticlockwise) and thus conclude in general

$$d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = d\vec{l} \cdot \vec{A}$$

$$\Rightarrow \int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) = \oint_S d\vec{l} \cdot \vec{A}$$ ($\because$ internal path cancel.)

III Integral vector calculus & Electrostatics

① Gradient theorem:

$$\vec{E} = -\vec{\nabla}\phi$$

$\Rightarrow$ $\phi(\vec{r}_1) - \phi(\vec{r}_2)$ is path independent

(or) $\int_a^b \vec{E}\cdot d\vec{l}$ is path independent.

② Divergence theorem: (Gauss's law)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$

$$\Rightarrow \oint_S d\vec{a}\cdot\vec{E} = 4\pi Q$$

③ Curl theorem:

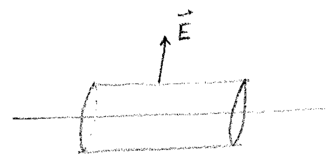
$$\vec{\nabla}\times\vec{E} = 0$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0$$

IV Gauss's law

$$\oint_S d\vec{a} \cdot \vec{E} = 4\pi Q$$

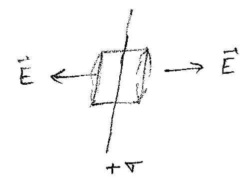
① Infinite line charge :



$$E \; 2\pi r h = 4\pi \lambda h$$

$$E = \frac{2\lambda}{r}$$

② Infinite surface charge



$$E \; 2A = 4\pi \sigma A$$

$$E = 2\pi \sigma$$

③ Uniform charge on a sphere

$$E \; 4\pi r^2 = 4\pi \; \frac{4\pi}{3} r^3 \rho$$

$$E = \begin{cases} \frac{4\pi}{3} \frac{a^3}{r^2} \rho & r > a \\ \frac{4\pi}{3} r \rho & r < a. \end{cases}$$