Electromagnetism H·W # (11)

(1) (a) Equation for the surface of a sphere can be written as

$$\xi(\vec{r}) = \sqrt{x^2 + y^2 + z^2} = a$$

where 'a' is the radius of the sphere.

Show that

$$\vec{\nabla}\, \xi(\vec{r}) = \hat{r}$$

(b) For a general surface $\xi(\vec{r}) - a = 0$ can you prove

$$\vec{\nabla}\, \xi(\vec{r}) = \hat{n}$$

where $\hat{n}$ is the normal unit vector on the surface. (I could <u>NOT</u> prove this.)

(2) Show that (ref: III-(6))

(i) $$\vec{\nabla}\left[\vec{P}_0 \cdot \vec{r}\right] = \vec{P}_0$$

(ii) $$\vec{\nabla}\left[\frac{\vec{P}_0 \cdot \hat{r}}{r^2}\right] = -\frac{1}{r^3}\left[3(\vec{P}_0 \cdot \hat{r})\,\hat{r} - \vec{P}_0\right]$$

(3) Show that (ref: IV-(5))

(i) $$\vec{\nabla} \times (\vec{M}_0 \times \vec{r}) = 2\vec{M}_0$$

(ii) $$\vec{\nabla} \times \left[\frac{(\vec{M}_0 \times \vec{r})}{r^3}\right] = \frac{2\vec{M}_0}{r^3}$$

4 (Refer problem 4.10 in Griffiths.)

A sphere of radius 'a' carries a polarization

$$\vec{P}(\vec{r}) = \alpha \vec{r}$$

where 'α' is a constant.

(a) Show that we can write

$$\vec{P}(\vec{r}) = \alpha \vec{r}\, \theta(a-r)$$

(b) Show that

$$\rho_{eff}(\vec{r}) = -3\alpha\, \theta(a-r) + \alpha r\, \delta(a-r)$$

(c) Choose $\vec{r}$ to be along $\hat{z}$ and for this choice show that

$$\phi(\vec{r}) = -6\pi\alpha \int_0^a r'^2 dr'\, I(r,r') + 2\pi\alpha a^3\, I(r,a)$$

where,

$$I(r,r') = \int_0^{\pi} \sin\theta' d\theta' \frac{1}{\left[r^2 + r'^2 - 2rr'\cos\theta'\right]^{1/2}}.$$

(Hint: $\int_0^{\infty} dx\, f(x)\, \theta(a-x) = \int_0^a dx\, f(x)$.)

(d) Evaluate :

$$I(r, r') = \int_0^{\pi} \sin\theta' d\theta' \frac{1}{\left[r^2 + r'^2 - 2rr'\cos\theta'\right]^{1/2}} = \begin{cases} \frac{2}{r'} & r < r' \\ \frac{2}{r} & r' < r \end{cases}$$

(e) Show that

$$\phi(\vec{r}) = \begin{cases} 2\pi\alpha\,(r^2 - a^2) & r < a \\ 0 & r > a \end{cases}$$

(Hint : For $r < a$, break the integral $\int_0^a dr' = \int_0^r dr' + \int_r^a dr'$.)

(f) Show that

$$\vec{E}(\vec{r}) = \begin{cases} -4\pi\alpha\,\vec{r} & r < a \\ 0 & r > a \end{cases}$$

(g) Interpret the result for $\vec{E}(\vec{r})$ for $r > a$ as the electric field due to the total charge inside $r < a$.

(h) Use Gauss's law to verify the result in (f).

⑤ (Refer: problem 6.7 in Griffiths.)

An infinitely long cylinder carries a uniform magnetization $\vec{M}_0$ parallel to its axis.

(a) Show that we can write

$$\vec{M}(\vec{r}) = M_0 \, \hat{z} \, \theta(a-s) \qquad s = \sqrt{x^2+y^2}$$

(b) Show that

$$\vec{J}_{eff}(\vec{r}) = M_0 \, (\hat{z} \times \hat{s}) \, \delta(a-s) = M_0 \, \hat{\phi} \, \delta(a-s)$$

(c) Show that

$$\vec{A}(\vec{r}) = \frac{M_0}{c} \, a \int_{-\infty}^{+\infty} dz' \int_0^{2\pi} d\phi' \frac{\hat{\phi}'}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') + (z-z')^2\right]^{1/2}}$$

(d) To my knowledge $\vec{A}(\vec{r})$ for this case can not be evaluated without invoking elliptic integrals. Starting from

$$\vec{B}(\vec{r}) = \frac{1}{c} \int d^3r' \, \frac{\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}$$

show that,

$$\vec{J}(\vec{r}') \times (\vec{r}-\vec{r}') = M_0 \, \delta(a-s') \left[(z-z') \, \hat{s}' + \{s' - s\sin(\phi-\phi')\} \, \hat{z}\right]$$

(e) Show that

$$\vec{B}(\vec{r}) = \frac{M_0}{c}\, a \int_0^{2\pi} d\phi' \int_{-\infty}^{+\infty} dz' \, \frac{\left[(z-z')\cos\phi' \,\hat{i} + (z-z')\sin\phi' \,\hat{j} + \{ a - s\sin(\phi-\phi') \} \,\hat{z} \right]}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') + (z-z')^2 \right]^{3/2}}$$

(f) Evaluate the integrals

$$\int_{-\infty}^{+\infty} dz' \, \frac{(z-z')}{\left[b^2 + (z-z')^2 \right]} = 0$$

and

$$\int_{-\infty}^{+\infty} dz' \, \frac{1}{\left[b^2 + (z-z')^2 \right]} = \frac{2}{b^2}$$

and using these show that

$$\vec{B}(\vec{r}) = \frac{2 M_0}{c}\, \hat{z}\, a \int_0^{2\pi} d\phi' \, \frac{\left[a - s\sin(\phi-\phi') \right]}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') \right]}$$

(g) Evaluate

$$\int_0^{2\pi} \sin\phi' \, d\phi' \, \frac{1}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') \right]} = 0$$

and thus conclude

$$\vec{B}(\vec{r}) = \frac{2 M_0}{c}\, \hat{z}\, a^2 \int_0^{2\pi} d\phi' \, \frac{1}{\left[s^2 + a^2 - 2as\cos(\phi-\phi') \right]}$$

(h) Use the result

$$\int d\phi \, \frac{1}{A + B\cos\phi} = \begin{cases} \dfrac{2}{\sqrt{A^2 - B^2}} \tan^{-1}\left[\dfrac{\sqrt{A^2-B^2}\,\tan\frac{\phi}{2}}{A+B} \right] & \text{for } A^2 > B^2 \\[2ex] \dfrac{1}{\sqrt{B^2 - A^2}} \ln\left[\dfrac{\sqrt{B^2-A^2}\,\tan\frac{\phi}{2} + (A+B)}{\sqrt{B^2-A^2}\,\tan\frac{\phi}{2} - (A+B)} \right] & \text{for } A^2 < B^2 \end{cases}$$

to show that

$$\vec{B}(\vec{r}) = \frac{2M_0}{c}\,\hat{z}\,\frac{2a^2}{s^2+a^2}\left[\tan^{-1}\frac{s^2+a^2}{(s-a)^2} + \tan^{-1}\frac{s^2+a^2}{(s+a)^2} \right]$$