Solution to prob. ④ – Hw # ⑪ (ref: prob 4.10 – Griffiths)

① $\vec{P}(\vec{r}) = \alpha \vec{r}\ \theta(a-r)$

② $$\rho_{eff}(\vec{r}) = -\vec{\nabla}\cdot\vec{P}(\vec{r})$$

$$= -\alpha\left[\vec{\nabla}\cdot\vec{r}\right]\theta(a-r) - \alpha\vec{r}\cdot\vec{\nabla}\theta(a-r)$$

$$= -3\alpha\ \theta(a-r) - \alpha\vec{r}\cdot\delta(a-r)(-1)\frac{\vec{r}}{r}$$

$$= -3\alpha\ \theta(a-r) + \alpha r\ \delta(a-r)$$

③ $$-\nabla^2\phi(\vec{r}) = 4\pi\rho(\vec{r})$$

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \int_0^\infty r'^2dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\varphi' \frac{\left[-3\alpha\ \theta(a-r') + \alpha r'\delta(a-r')\right]}{\left[r^2+r'^2-2rr'\cos\theta'\right]^{1/2}}$$

($\vec{r}$ chosen along $\hat{z}$)

$$= -6\pi\alpha\int_0^a r'^2dr'\int_0^\pi \sin\theta' d\theta' \frac{1}{\left[r^2+r'^2-2rr'\cos\theta'\right]^{1/2}} + 2\pi\alpha a^3\int_0^\pi \sin\theta' d\theta' \frac{1}{\left[r^2+a^2-2ar\cos\theta'\right]^{1/2}}$$

$$= -6\pi\alpha\int_0^a r'^2dr'\ I(r,r') + 2\pi\alpha a^3\ I(r,a)$$

where,

$$I(r,r') = \int_0^\pi \sin\theta' d\theta' \frac{1}{\left[r^2+r'^2-2rr'\cos\theta'\right]^{1/2}}$$

4 $$I(r,r') = \int_0^{\pi} \sin\theta' \, d\theta' \frac{1}{[r^2+r'^2-2rr'\cos\theta']^{1/2}}$$

$$\cos\theta' = t \qquad -\sin\theta' d\theta' = dt$$

$$= \int_{-1}^{+1} dt \frac{1}{[r^2+r'^2-2rr't]^{1/2}}$$

$$r^2+r'^2-2rr't = y$$
$$-2rr'\,dt = dy$$

$$= \int_{(r+r')^2}^{(r-r')^2} -\frac{dy}{2rr'} \frac{1}{\sqrt{y}}$$

$$\frac{y^{-\frac{1}{2}+1}}{\frac{1}{2}} = 2\sqrt{y}$$

$$= \frac{1}{2rr'} \int_{(r-r')^2}^{(r+r')^2} \frac{dy}{\sqrt{y}}$$

$$= \frac{1}{rr'} \sqrt{y} \Big|_{(r-r')^2}^{(r+r')^2}$$

$$= \frac{1}{rr'}[r+r'-(r-r')] \qquad r'<r$$

$$\frac{1}{rr'}[r+r'-(r'-r)] \qquad r<r'$$

$$= \begin{cases} \frac{2}{r} & r'<r \\ \frac{2}{r'} & r<r' \end{cases}$$

5 $$\phi(\vec{r}) = \begin{cases} -6\pi\alpha \int_0^r r'^2 dr' \, I(r,r') - 6\pi\alpha \int_r^a r'^2 dr' \, I(r,r') + 2\pi\alpha a^3 I(r,a) & r<a \\ -6\pi\alpha \int_0^a r'^2 dr' \, I(r,r') + 2\pi\alpha a^3 I(r,a) & r>a \end{cases}$$

⑥ $$\phi(\vec r) = \begin{cases} -6\pi\alpha \int_0^r r'^2 dr' \frac{2}{r} - 6\pi\alpha \int_r^a r'^2 dr' \frac{2}{r'} + 2\pi\alpha a^3 \frac{2}{a} & r < a \\ -6\pi\alpha \int_0^a r'^2 dr' \frac{2}{r} + 2\pi\alpha a^3 \frac{2}{r} & r > a \end{cases}$$

$$= \begin{cases} -4\pi\alpha r^2 - 6\pi\alpha(a^2 - r^2) + 4\pi\alpha a^2 & r < a \\ -4\pi\alpha \frac{a^3}{r} + 4\pi\alpha \frac{a^3}{r} & r > a \end{cases}$$

$$= \begin{cases} 2\pi\alpha r^2 - 2\pi\alpha a^2 & r < a \\ 0 & r > a. \end{cases}$$

⑦ $$\vec E(\vec r) = -\vec\nabla \phi(\vec r)$$

$$= \begin{cases} -2\pi\alpha \vec\nabla(r^2) & r < a \\ 0 & r > a. \end{cases}$$

$$= \begin{cases} -4\pi\alpha \vec r & r < a \\ 0 & r > a \end{cases}$$

⑧ $$\vec\nabla \cdot \vec E = -4\pi \vec\nabla \cdot \vec P$$

$$\oint d\vec a \cdot \vec E = -4\pi \oint d\vec a \cdot \vec P \qquad (r < a)$$

$$4\pi r^2 E(r) = -4\pi \, 4\pi r^2 \alpha r$$

$$E(r) = -4\pi\alpha \vec r \quad \text{for} \quad r < a$$