Meeting No. ①

I ① Vector : $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x^i$ $\quad i = 1, 2, \ldots, N$

$x^i \longrightarrow$ components

Matrix: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{ij}$

② $A = a_{ij} \Rightarrow A^T = a_{ji}$

③ Identity $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \delta_{ij}$ — Kronecker delta

II ① Transformations

$$x' = f(x, y)$$
$$y' = g(x, y)$$

eg:
$$x' = x^2 + y^9 + x^4 y^7$$
$$y' = x^{13} + y^{23}$$

② Linear transformations

$$\bar{x}_1 = a_{11} x_1 + a_{12} x_2 + b_1$$

$$\bar{x}_2 = a_{21} x_1 + a_{22} x_2 + b_2.$$

(3) Summation convention

$$a_{11} x_1 + a_{12} x_2 = \sum_{j=1}^{2} a^1{}_j x^j = a^1{}_j x^j$$

The linear transformations using the summation conventions reads

$$\bar{x}^i = L^i{}_j x^j + b^i$$

$i \rightarrow$ free index
$j \rightarrow$ dummy index (summed over)

(4) Evaluate the components of

(i) $\frac{\partial x^i}{\partial x^j} =$

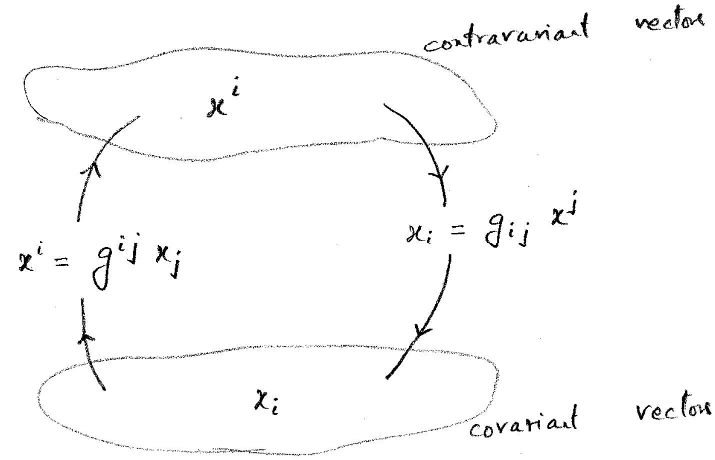
(ii) $\frac{\partial \bar{x}^i}{\partial x^j} =$

(5) Evaluate

$$L^i{}_j \delta^j{}_k = \sum_j L^i{}_j \delta^j{}_k = ?$$

This indicates how Kronecker delta simplifies equations.

IV 1 Contravariant and covariant vectors and the metric



distance $\equiv$ $x^i x_i = x^i g_{ij}\, x^j$

2 Properties of g_{ij}

(i) g_{ij} is symmetric ($g_{ij} = g_{ji}$).

(ii) $g^{ij} g_{jk} = \delta^i{}_k$ (upper and lower are inverses of each other).

3 Contravariant vectors transform like position vector (by definition)

$$\bar{x}^i = L^i{}_j\, x^j$$

4. Prove: g_{ij} is symmetric.

$$x^i g_{ij} x^j = \sum_i \sum_j g_{ij} x^i x^j$$

$$= \sum_j \sum_i g_{ji} x^j x^i$$

$$= \sum_i \sum_j g_{ji} x^i x^j$$

$$= x^i g_{ji} x^j$$

$\leftarrow$ A simple trick involved in handling dummy indices.

$$\Rightarrow \quad g_{ij} = g_{ji}$$

5. Prove: $g^{ij} g_{jk} = \delta^i{}_k$

$$x^i = g^{ij} x_j$$

$$= g^{ij} g_{jk} x^k$$

$$\Rightarrow \quad g^{ij} g_{jk} = \delta^i{}_k$$

6. Def: that covariant vectors transform as

$$x_i = M_i{}^j \bar{x}_j$$

(Note that bar is on RHS. This is defined so that $\vec{\nabla}$ transforms like a covariant).

7. Observe that :

(i) $L^i{}_j = \dfrac{\partial \bar{x}^i}{\partial x^j}$

(ii) $M_i{}^j = \dfrac{\partial x_i}{\partial \bar{x}_j}$

(iii) $g^{ij} = \dfrac{\partial x^i}{\partial x_j}$

(iv) $g_{ij} = \dfrac{\partial x_i}{\partial x^j}$

8. Also :

(i) $\dfrac{\partial \bar{x}^i}{\partial x^j} \dfrac{\partial x^k}{\partial \bar{x}^i} = \delta_j{}^k$

(ii) $\dfrac{\partial x^i}{\partial \bar{x}^j} \dfrac{\partial \bar{x}^k}{\partial x^i} = \delta_j{}^k$

(iii) $M = g\, L^{-1} g^{-1}$ ← (THINK and check)

9. Show : that gradiant operator $\left(\nabla_i = \dfrac{\partial}{\partial x^i} \right)$ transforms like a covariant vector

$$\frac{\partial}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \frac{\partial}{\partial x^j}$$

$$= M_i{}^j \frac{\partial}{\partial x^j}.$$

V ① Transformations that keep the distance invariant

$$\bar{x}^i = L^i{}_j \, x^j$$

② $$x^i x_i = \bar{x}^i \bar{x}_i$$
$$= \bar{x}^i g_{ij} \bar{x}^j$$
$$= L^i{}_m x^m g_{ij} L^j{}_n x^n$$
$$= L^i{}_m g_{ij} L^j{}_n x^m x^n$$

$$\Rightarrow \quad L^i{}_m g_{ij} L^j{}_n = g_{mn}$$

③ Scalars (0-rank tensor) $\qquad f(a^i) = f(L^i{}_j a^j)$

Vector (1-rank tensor) $\qquad \bar{a}^i = L^i{}_m a^m$

2-rank tensor $\qquad \bar{a}^{ij} = L^i{}_m L^j{}_n a^{mn}.$

$$\bar{a}^{ij}{}_{pqr} = L^i{}_m L^j{}_n M_p{}^u M_q{}^v M_r{}^w a^{mn}{}_{uvw}$$

4. Addition of tensors:

$$A^{ijk}{}_{l} + B^{ijk}{}_{l} = C^{ijk}{}_{l}$$

Note the indices are same in each term

5. Levi-Civita symbol

$$\epsilon_{ijk}$$

$$\epsilon_{123} = +1$$