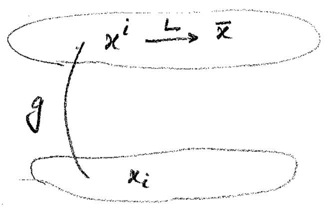
Meeting No. 2

I) 1. Summary of meeting no. 1 on transformations and metric



$$x^i = g^{ij} x_j$$

$$\bar{x}^i = L^i{}_j \, x^j$$

$$\bar{x}_i = M_i{}^j \, x_j$$

2. Lorentz transformation will be a transformation in (3+1) space-time dimensions. with metric

$$g^{\alpha\beta} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

3. Notation :

Greek $\rightarrow$ $\alpha, \beta, \gamma, \ldots$ run from $0, 1, 2, 3$

Latin $\rightarrow$ $i, j, k, \ldots$ run from $1, 2, 3$

eg: $\quad x^\mu \equiv (x^0, x^i) \equiv (x^0, x^1, x^2, x^3)$

(4) Position 4-vector

$$x^\mu \equiv (ct\,,\ \vec{x})$$

$$\partial_\mu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\ \vec{\nabla}\right)$$

(5) Show:

(i) $x_\mu = g_{\mu\nu}\, x^\nu \equiv (ct\,, -\vec{x})$

(ii) $\partial^\mu = g^{\mu\nu}\partial_\nu \equiv \left(\frac{1}{c}\frac{\partial}{\partial t}\,, -\vec{\nabla}\right)$

(iii) $\Delta x^\mu\, \Delta x_\nu = c^2\,\Delta t^2 - \Delta\vec{x}\cdot\Delta\vec{x}$

(iv) $\partial_\mu\,\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$

(v) $x_\mu\,\partial^\mu = t\,\frac{\partial}{\partial t} - \vec{x}\cdot\vec{\nabla}$

Lorentz transformations

II (1) Lorentz transformations are the set of transformations of the kind $L^{\mu}{}_{\nu}$ acting on 4-vectors $x^{\mu} \equiv (ct, \vec{x})$

$$\bar{x}^{\mu} = L^{\mu}{}_{\nu}\, x^{\nu}$$

such that they leave the quantity

$$\Delta x^{\mu} \Delta x_{\nu} = c^2 \Delta t^2 - \Delta \vec{x} \cdot \Delta \vec{x}$$

invariant.

(2)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \overbrace{\begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}}^{L} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(3) Show that:

(i) $L\, g\, L^{T} = g$

(ii) $[\det L]^2 = 1$

$\det(L g L^{T}) = \det g$
$\det(L^{T} L g) = \det g$
$\det(L^{T} L) \det g = \det g$ $(\because g$ is dia$)$
$\det(L^{T} L) = 1$
$[\det L]^2 = 1$

(4) Rotations in 3-D space leaves the quantity $x^i x_i = x^2 + y^2 + z^2$ invariant. $\left(g^{ij} = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \right)$

Show that:

(i) $R\, R^{T} = 1$

(ii) $[\det. R]^2 = 1$

(4) <u>Show</u> that rotation matrix has only 3 independent components

$R R^T = 1$

$\Rightarrow \quad R_{ij} R_{jk} = \delta_{ik}$

— diagonal elements of RHS give 3 eqns.

— Other elements reproduce duplicate equations $= \frac{6}{2}$ eqns.

$\therefore$ Independent components $= \underbrace{9}_{\text{original no. of comp.}} - 3 - \frac{6}{2} = 3$

(5) <u>Show</u> that the Lorentz matrix has only 6 independent components.

$= 16 - 4 - \frac{12}{2} = 6$

(6) If $\quad ct \rightarrow ict$

$g^{\alpha\beta} \rightarrow (+1, +1, +1, +1)$

then L satisfies

$L L^T = 1$

instead of the earlier

$L g L^T = g$.

(7) Illustrate any rotation matrix can be written as the product of 3 independent rotations given by

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} ; \quad R_y = \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} ; \quad R_z = \begin{pmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ie) $R(\theta_x, \theta_y, \theta_z) = R_x R_y R_z$

(8) Illustrate any Lorentz matrix can be written as the product of 3 boost and 3 rotation matrices.

$$K_x = \begin{pmatrix} \gamma_x & -\beta\gamma_x & 0 & 0 \\ -\beta\gamma_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad K_y = \begin{pmatrix} \gamma_y & 0 & -\beta_y\gamma_y & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad K_z = \begin{pmatrix} \gamma_z & 0 & 0 & -\beta_z\gamma_z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta_z\gamma_z & 0 & 0 & \gamma_z \end{pmatrix}$$

$\beta = \frac{v}{c}$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_x & \sin\theta_x \\ 0 & 0 & -\sin\theta_x & \cos\theta_x \end{pmatrix} \quad R_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_y & 0 & \sin\theta_y \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta_y & 0 & \cos\theta_y \end{pmatrix} \quad R_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_z & \sin\theta_z & 0 \\ 0 & -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(9) (i) Rotations in 3-D donot commute: Take an object that is not rotationally symmetric (a book)

(a) → rotate by $+90^\circ$ about x-axis
(b) → rotate by $+90^\circ$ about y-axis
(c) → rotate by -90° about x-axis
(d) → rotate by -90° about y-axis

$$R_y^{-1} R_x^{-1} R_y R_x (\text{object}) \neq \text{object}$$

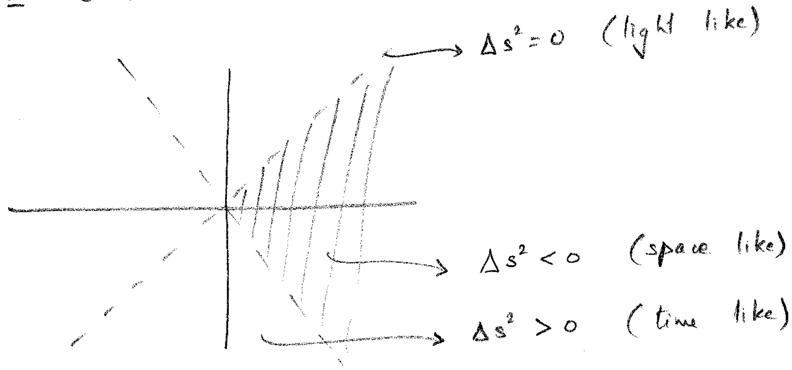
(ie) $R_y R_x \neq R_x R_y$

(ii) Similarly Lorentz boosts do not commute.

III Space-time diagrams :

① $$\Delta s^2 = c^2 \Delta t^2 - \Delta \vec{x} \cdot \Delta \vec{x}$$

'event' $= x_1^\mu - x_2^\mu$



② (i) Δs^2 is invariant under Lorentz transformations. Thus the property of a vector being space, time or light like is invariant under Lorentz transformations

(ii) A space-like event can be made simultaneous $(\Delta t = 0)$ in a particular frame.

(iii) A time-like event can be made such that $\Delta x = 0$, (ie) it is purely time-like in a particular frame.

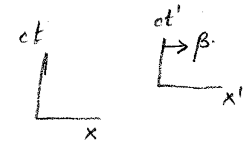
③ All physical processes are causal.

$\Rightarrow$ all physical quantities should be time-like.

④ World line : Path traced by a particle.

World lines are always with in the light cone.

IV Time dilation



① $$\begin{pmatrix} c\,\Delta t' \\ \Delta x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} c\,\Delta t \\ \Delta x \end{pmatrix}$$

$$\begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \underbrace{\begin{pmatrix} c\,\Delta t' \\ \underbrace{\Delta x'}_{=0} \end{pmatrix}}_{\text{rest frame of the particle}} = \begin{pmatrix} c\,\Delta t \\ \Delta x \end{pmatrix}$$

→ The primed frame is the rest frame of a point particle.

$$\gamma\,\Delta t' = \Delta t \qquad \text{(dilation)}$$

② $$c^2\,\Delta t'^2 - \underbrace{\Delta x'^2}_{=0} = c^2\,\Delta t^2 - \Delta x^2$$

$$c^2\,\Delta t'^2 = c^2\,\Delta t^2 - \Delta x^2 .$$

$$\Rightarrow \quad \Delta t' > \Delta t \qquad \text{(dilation}$$

③ Proper time :

$$\underbrace{\{c^2\,\Delta t'^2\}_{\text{rest f.}}}_{\text{proper time}} = \{c^2\,\Delta t^2 \cdot \Delta x_p^2\}_{\text{another frame.}}$$

$$= c^2\,\Delta t^2 \left\{1 - \frac{1}{c^2}\frac{\Delta x_p^2}{\Delta t^2}\right\}$$

$$= c^2\,\Delta t^2\,(1 - \beta_p^2)$$

→ proper time is an invariant quantity.

V 4-Position and 4-velocity

① $x^\mu = (ct, \vec{x})$

② 4-velocity

$$u^\alpha \equiv \frac{dx^\alpha}{d\tau} = \frac{d}{d\tau}(ct, \vec{x})$$

$$= \left(c\frac{dt}{d\tau}, \frac{dt}{d\tau}\vec{v}\right)$$

$$= (c\gamma, \beta\gamma c)$$

③ Show:

$u^\alpha u_\alpha = +1$.

V Velocity addition

$\lfloor 1 \rightarrow \beta_1$

$\lfloor$ real 0 $\qquad \lfloor 2 \rightarrow \beta_2$.

$$u_1^\alpha = L_{10}\, u_0^\alpha$$

$$u_2^\alpha = L_{20}\, u_0^\alpha$$

$$u_2^\alpha = L_{21}\, u_1^\alpha$$

$$\therefore \quad L_{20}\, u_0 = L_{21} L_{10}\, u_0$$

$$L_{20} = L_{21} L_{10}$$

$$L_{21} = L_{20} L_{10}^{-1}$$

$$\begin{pmatrix} \gamma_{12} & \beta_{12}\gamma_{12} \\ \beta_{12}\gamma_{12} & \gamma_{12} \end{pmatrix} = \begin{pmatrix} \gamma_2 & \beta_2\gamma_2 \\ \beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_1\gamma_2(1-\beta_1\beta_2) & \gamma_1\gamma_2(\beta_2-\beta_1) \\ \gamma_1\gamma_2(\beta_2-\beta_1) & \gamma_1\gamma_2(1-\beta_1\beta_2) \end{pmatrix}$$

$$\Rightarrow \quad \gamma_{12} = \gamma_1\gamma_2(1-\beta_1\beta_2)$$

$$\beta_{12}\gamma_{12} = \gamma_1\gamma_2(\beta_2-\beta_1)$$

$$\Rightarrow \quad \beta_{12} = \frac{\beta_2-\beta_1}{1-\beta_1\beta_2}$$