Meeting No. (4)

I Motion in 1-D with constant acceleration

refer: Wolfgang Rindler, Relativity page 73-75

(1) $u^\alpha = (c\gamma, c\beta\gamma, 0, 0)$

$$a^\alpha = \frac{d}{d\tau} u^\alpha = \left(c\gamma \frac{d\gamma}{dt}, c\gamma \frac{d}{dt}(\beta\gamma) \right)$$

(2) $\frac{d\beta}{dt} \equiv$ physical acceleration experienced in a particular frame.

(3) Let $\beta = \tanh\theta$

$$\frac{d\beta}{dt} = (1 - \tanh^2\theta)\frac{d\theta}{dt}$$

$$\gamma^2 \frac{d\beta}{dt} = \frac{d\theta}{dt}$$

(4) S_1 and S_2 frame moving along x-axis.

$$\beta_R = \frac{\beta_1 - \beta_2}{1 - \beta_1\beta_2}$$

$$\theta_R(\beta_R) = \theta_1(\beta_1) - \theta_2(\beta_2)$$

$$0 = \frac{d\theta_1}{dt_1} - \frac{d\theta_2}{dt_1}$$

($\because$ β_1 and β_2 are particle velocities they have time dependence. β_R does not have explicit time dependence.)

(5) $$\frac{d\theta_1}{dt_1} = \frac{dt_2}{dt_1}\frac{d\theta_2}{dt_2}$$

6 Using 3 in 5

$$r_1^2 \frac{d\beta_1}{dt_1} = \frac{dt_2}{dt_1} r_2^2 \frac{d\beta_2}{dt_2}$$

7 $$dt_1^2 (1 - \beta_1^2) = dt_2^2 (1 - \beta_2^2)$$

$$\frac{dt_2}{dt_1} = \frac{r_2}{r_1}$$

8 Using 7 in 6

$$r_1^3 \frac{d\beta_1}{dt_1} = r_2^3 \frac{d\beta_2}{dt_2}$$ $\longrightarrow$ Transformation of physical acceleration in relativity

9 <u>Show</u>: Non-relativistic limit ($\beta << 1$) leads to

$$\frac{dv_1}{dt_1} = \frac{dv_2}{dt_2}$$

This is the transformation of acceleration under Galilean transformation.

(10) Consider a particle moving along the x-axis with a constant acceleration $\alpha \left(= c^2 \frac{d^2 x}{dt^2} \right)$.

Then we have in an arbitrary frame

$$\gamma^3 \frac{d\beta}{dt} = \frac{\alpha}{c} = \text{constant}$$

$$\frac{d}{dt}(\beta\gamma) = \frac{\alpha}{c} = \text{constant}$$

(11) $$\int_{t=0}^{t=t} dt \frac{d}{dt}(\beta\gamma) = \int_{t=0}^{t=t} dt \frac{\alpha}{c}$$

$$\{\beta\gamma\}_{t=t} - \{\beta\gamma\}_{t=0} = \frac{\alpha}{c}(t-0)$$

(12) Let $\{\beta\}_{t=0} = 0$

$$\beta\gamma = \frac{\alpha}{c} t$$

$$\frac{\beta^2}{1-\beta^2} = \frac{\alpha^2}{c^2} t^2$$

$$\beta^2 = (1-\beta^2) \frac{\alpha^2}{c^2} t^2$$

$$\beta^2 = \frac{\alpha^2 t^2 / c^2}{1 + \frac{\alpha^2 t^2}{c^2}}$$

$$\frac{dx}{dt} = \frac{\alpha t}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}}$$

13

$$1 + \frac{\alpha^2 t^2}{c^2} = \mu$$

$$2 \frac{\alpha^2}{c^2} t \, dt = d\mu$$

$$\int_{t=0}^{t=t} dt \, \frac{dx}{dt} = \int_{t=0}^{t=t} dt \, \frac{\alpha t}{\sqrt{1 + \frac{\alpha^2 t^2}{c^2}}}$$

$$x(t) - x(0) = \int_{\mu=1}^{1+\frac{\alpha^2 t^2}{c^4}} \frac{d\mu}{2 \frac{\alpha^2}{c^2}} \; \alpha \; \frac{1}{\sqrt{\mu}}.$$

$$= \frac{1}{2} \frac{c^2}{\alpha} \frac{1}{\left(\frac{1}{2}\right)} \left\{ \sqrt{\mu} \right\}_{\mu=1}^{\mu = 1 + \frac{\alpha^2}{c^4} t^2}.$$

$$x(t) - x(0) = \frac{c^2}{\alpha} \left[\sqrt{1 + \frac{\alpha^2 t^2}{c^4}} - 1 \right]$$

14 let $x(t=0) = \frac{c^2}{\alpha}$

$$\Rightarrow \quad x(t) = \frac{c^2}{\alpha} \sqrt{1 + \frac{\alpha^2 t^2}{c^4}}$$

$$x^2 = \frac{c^4}{\alpha^2} \left(1 + \frac{\alpha^2 t^2}{c^2} \right)$$

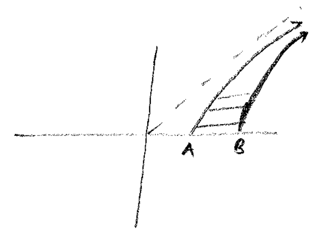
$$x^2 = \frac{c^4}{\alpha^2} + c^2 t^2$$

$$x^2 - c^2 t^2 = \frac{c^4}{\alpha^2} \longrightarrow \text{eq}^n. \text{ of hyperbola.}$$

II. Length contraction

1. Using I we have

$$x^2 - c^2t^2 = \frac{c^4}{\alpha^2}$$



2. Define length to be the distance between two physical point particles A and B (A - 1 particle, B - 2 particle in 1)

$$x_B^2 - c^2t^2 = \frac{c^4}{\alpha_B^2}$$

$$x_A^2 - c^2t^2 = \frac{c^4}{\alpha_A^2}$$

3. $$x_B^2 - x_A^2 = \frac{c^4}{\alpha_B^2} - \frac{c^4}{\alpha_A^2}$$

4. $t=0$, AB is maximum, they are at rest.

finite t, AB is less, they are moving.

Conclude length contraction.

(ref: prob 3.23 Rindler)

III Collisions

Example 1 : (Decay of particle)

① $\underset{\text{(at rest)}}{\pi} \longrightarrow \mu + \nu$

ν: mass $= 0$

② $$P_\pi^\alpha = P_\mu^\alpha + P_\nu^\alpha$$

$$(m_\pi c, 0) = \left(\frac{E_\mu}{c}, \vec{P}_\mu\right) + \left(|\vec{P}_\nu|, \vec{P}_\nu\right)$$

③ $$P_\pi^2 = P_\mu^2 + P_\nu^2 + 2 P_\mu P_\nu$$

$$m_\pi^2 = m_\mu^2 + 0 + 2\frac{E_\mu}{c}|\vec{P}_\nu| - 2\vec{P}_\mu \cdot \vec{P}_\nu$$

$\downarrow$

more work

④ $$P_\pi^\alpha - P_\mu^\alpha = P_\nu^\alpha$$

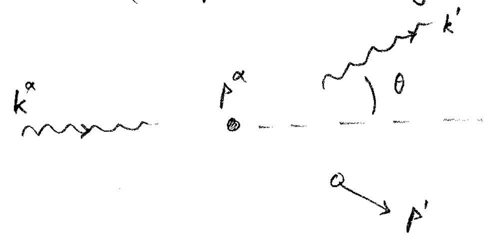
$$P_\pi^2 + P_\mu^2 - 2 P_\pi^\alpha P_{\mu\alpha} = 0$$

$$m_\pi^2 + m_\mu^2 - 2 m_\pi c \frac{E_\mu}{c} + 0 = 0$$

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2 m_\pi}$$

Example 2 (Compton scattering)

1



$$k^\alpha = (\hbar k, \hbar \vec{k})$$
$$p^\alpha = (mc, 0)$$
$$k'^\alpha = (\hbar k', \hbar \vec{k}')$$
$$p'^\alpha = \left(\frac{E}{c}, \vec{p}\right)$$

2
$$p + k = p' + k'$$
$$p + k - k' = p'$$

3
$$\cancel{p^2} + (k'-k)^2 - 2p(k'-k) = \cancel{p'^2}$$
$$(k'-k)^2 = 2p(k'-k)$$
$$\underbrace{k'^2}_{=0} + \underbrace{k^2}_{=0} - 2k'_\alpha k^\alpha = 2mc(k'-k)$$
$$-(\hbar k \hbar k' - \hbar^2 \vec{k}\cdot\vec{k}') = mc(\hbar k' - \hbar k)$$
$$-\hbar^2 k k' (1 - \cos\theta) = mc\hbar(k'-k)$$
$$-\hbar \frac{4\pi^2}{\lambda\lambda'}(1-\cos\theta) = mc\, 2\pi \frac{\lambda - \lambda'}{\lambda\lambda'}$$
$$\lambda' - \lambda = \frac{2\pi\hbar}{mc}(1 - \cos\theta)$$

$$\left\{\begin{array}{c}\text{compton}\\ \text{wavelength}\end{array}\right\} = \frac{\pi\hbar}{mc}$$

Example 3 : (Pair creation)

1

2 $$p^\alpha = q_1^\alpha + q_2^\alpha$$

$$p - q_1 = q_2$$

3 $$\underbrace{p^2}_{=0} + \cancel{q_1^2} - 2 p^\alpha q_{1\alpha} = \cancel{q_2^2}$$

$$2 p_\alpha q_1^\alpha = 0$$

$$\hbar k \frac{E_1}{c} - \hbar \vec{k} \cdot \vec{q}_1 = 0$$

$$\hbar k \frac{E_1}{c} = \hbar k |q_1| \cos\theta$$

$$\cos\theta = \frac{E_1}{|q_1| c} = \sqrt{\frac{q_1^2 c^2 + m^2 c^4}{q_1^2 c^2}} > 1$$

$\Rightarrow$ one photon cannot create $e^- - e^+$ pair.

4 Give a physical arguement. Goto the rest frame of the final particles.

Terminologies

1. Zero-momentum frame (center of momentum frame)

2. Threshold energies.

$$A + \underset{\substack{\downarrow \\ \text{(at rest)}}}{B} \longrightarrow \sum_i C_i$$

$$p_A^2 + p_B^2 + 2 m_B c \frac{E_A}{c} = \sum_i m_i^2 c^2 + \left\{ \begin{array}{l} \text{factor dep. on momentum} \\ \text{of } C_i \end{array} \right\}.$$

$$\Rightarrow \quad E_A > \frac{\left(\sum_i m_i^2 c^2\right) - m_A^2 c^2 - m_B^2 c^2}{2 m_B}$$