Meeting No. ⑤

I 4 - acceleration

① $$a^\alpha = \frac{du^\alpha}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt}\left(c\gamma, \vec{v}\gamma\right)$$
$$= \gamma\left(c\frac{d\gamma}{dt}, \vec{a}\gamma + \vec{v}\frac{d\gamma}{dt}\right)$$

② $$u^\alpha = \gamma(c, \vec{v})$$

③ Using $a^\alpha u_\alpha = 0$

$$c^2\frac{d\gamma}{dt} - \vec{v}\cdot\left[\vec{a}\gamma + \vec{v}\frac{d\gamma}{dt}\right] = 0$$

$$\left(1 - \frac{v^2}{c^2}\right)\frac{d\gamma}{dt} = \frac{(\vec{v}\cdot\vec{a})}{c^2}\gamma$$

$$\frac{d\gamma}{dt} = \gamma^3\frac{(\vec{v}\cdot\vec{a})}{c^2}$$

④ Using ③ in ①

$$a^\alpha = \gamma\left(c\gamma^3\frac{(\vec{v}\cdot\vec{a})}{c^2}, \vec{a}\gamma + \vec{v}\gamma^3\frac{(\vec{v}\cdot\vec{a})}{c^2}\right)$$
$$= \gamma^2\left(c\gamma^2\frac{(\vec{v}\cdot\vec{a})}{c^2}, \vec{a} + \vec{v}\gamma^2\frac{(\vec{v}\cdot\vec{a})}{c^2}\right)$$

⑤ Prope acceleration (when $\vec{v} = 0$)

$$a^\alpha = (0, \vec{a}_0)$$

⑤ $$a^\alpha a_\alpha = \gamma^4\left[c^2 \gamma^4 \frac{(\vec{v}\cdot\vec{a})^2}{c^4} - \left\{\vec{a} + \vec{v}\,\gamma^2 \frac{(\vec{v}\cdot\vec{a})}{c^2}\right\}^2\right]$$

$$\Big\downarrow \text{H.W}$$

$$= -\gamma^4\left[a^2 + \gamma^2 \frac{(\vec{v}\cdot\vec{a})^2}{c^2}\right]$$

$$= -\gamma^6\left[a^2 - \frac{(\vec{v}\times\vec{a})^2}{c^2}\right]$$

⑥ In a particular frame

(i) if $\vec{a} \parallel \vec{v}$ (linear motion)

$$|\vec{a}_0| = \gamma^3 |\vec{a}|$$

(ii) if $\vec{a} \perp \vec{v}$ (circular motion)

$$|\vec{a}_0| = \gamma^2 |\vec{a}|$$

⑦ $$(\vec{A}\times\vec{B})^2 = \epsilon^{ijk} A_j B_k \epsilon^{imn} A_m B_n$$

$$= (\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}) A_j B_k A_m B_n$$

$$= A^2 B^2 - (\vec{A}\cdot\vec{B})^2.$$

II 4 – Force

① Define → $\dfrac{d\vec{p}}{dt} = \vec{F}$ — relativistic 3-force.

Define → $\dfrac{dp^{\alpha}}{d\tau} = F^{\alpha}$ — 4-force, (Lorentz force obeys this)

② $$F^{\alpha} = \frac{dt}{d\tau}\frac{dp^{\alpha}}{dt}$$

$$= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$$

$$= \gamma\left(\frac{1}{c}\frac{dE}{dt}, \vec{F}\right)$$

③ Show that $F^{\alpha} u_{\alpha} = 0$

④ $u_{\alpha} = \gamma(c, \vec{v})$

$$\Rightarrow \quad \frac{dE}{dt} = \vec{F}\cdot\vec{v}$$

⑤ Using ④ in ②

$$F^{\alpha} = \gamma\left(\frac{(\vec{F}\cdot\vec{v})}{c}, \vec{F}\right)$$

6 $$\vec{F} = \frac{d\vec{P}}{dt}$$

$$= \frac{d}{dt}(m \vec{v}\, r)$$

$$= m \vec{a}\, r + m \vec{v} \frac{dr}{dt}$$

7 $$E = mc^2 r$$

$$\frac{dE}{dt} = mc^2 \frac{dr}{dt}$$

$$F \cdot \vec{v} = mc^2 \frac{dr}{dt} \quad \text{(using 4)}$$

8 Using 7 in 6

$$\vec{F} = m \vec{a}\, r + m \vec{v} \frac{\vec{F} \cdot \vec{v}}{mc^2}$$

$$= m \vec{a}\, r + \vec{V} \frac{(\vec{F} \cdot \vec{v})}{c^2}$$

9 Note that $\vec{F}$ and $\vec{a}$ need not be parallel.

(10) Write vector in terms of parallel and perpendicular component along $\vec{v}$.

$$\vec{F} = \vec{F}_{\parallel} + \vec{F}_{\perp}$$

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$

(11) $$\vec{F} = \gamma m \vec{a} + \vec{v} \frac{(\vec{F} \cdot \vec{v})}{c^2}$$

$$\vec{F}_{\parallel} + \vec{F}_{\perp} = \gamma m (\vec{a}_{\parallel} + \vec{a}_{\perp}) + \frac{v^2}{c^2} \vec{F}_{\parallel}$$

$$\Rightarrow \boxed{\vec{F}_{\parallel} = \gamma^3 m \vec{a}_{\parallel}} \qquad \boxed{\vec{F}_{\perp} = \gamma m \vec{a}_{\perp}}$$

III Motion in 1-D with constant proper acceleration

① In I we saw

$$a^\alpha a_\alpha = -\vec{a}_0 \cdot \vec{a}_0 = -\gamma^4 \left[a^2 + \gamma^2 \frac{(\vec{v}\cdot\vec{a})^2}{c^2} \right]$$

$$= -\gamma^6 \left[a^2 - \frac{(\vec{v}\times\vec{a})^2}{c^2} \right]$$

② For motion in 1 D with $\vec{a} \parallel \vec{v}$ we have.

$$a^\alpha a_\alpha = -\vec{a}_0 \cdot \vec{a}_0 = -\gamma^6 a^2 = \text{const}$$

$$\Rightarrow \quad \gamma^3 a = a_0$$

$$\Rightarrow \quad \gamma^3 \frac{d\beta}{dt} = \frac{a_0}{c}$$

$$\Rightarrow \quad \frac{d}{dt}(\beta\gamma) = \frac{a_0}{c}$$

③ $$\int_{t=0}^{t=t} dt \frac{d}{dt}(\beta\gamma) = \int_{t=0}^{t=t} dt \frac{a_0}{c}.$$

$$\{\beta\gamma\}_{t=t} - \{\beta\gamma\}_{t=0} = \frac{a_0}{c}(t-0)$$

④ Let $\beta(t=0) = 0$

$$\Rightarrow \quad \beta\gamma = \frac{a_0}{c} t$$

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$$\beta^2 \gamma^2 = \frac{a_0^2}{c^2} t^2 .$$

$$\beta^2 = \frac{\frac{a_0^2 t^2}{c^2}}{1 + \frac{a_0^2 t^2}{c^2}}$$

$$\frac{dx}{dt} = \frac{a_0 t}{\sqrt{1 + \frac{a_0^2 t^2}{c^2}}}$$

$$1 + \frac{a_0^2 t^2}{c^2} = \mu .$$

$$\frac{a_0^2}{c^2} 2t\,dt = d\mu .$$

$$t\,dt = \frac{c^2}{2 a_0^2} d\mu .$$

$$x(t) - x(0) = \int_{t=0}^{t=t} dt \, \frac{a_0 t}{\sqrt{1 + \frac{a_0^2 t^2}{c^2}}}$$

$$= \int_{\mu=1}^{\mu = 1 + \frac{a_0^2 t^2}{c^2}} \frac{c^2}{2 a_0^2} d\mu \, \frac{a_0}{\sqrt{\mu}}$$

$$= \frac{c^2}{a_0} \left[\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right]$$

6 Let $x(0) = \frac{c^2}{a_0}$

$$x(t)^2 = \frac{c^4}{a_0^2} \left(1 + \frac{a_0^2 t^2}{c^2} \right)$$

$$x(t)^2 - c^2 t^2 = \frac{c^4}{a_0^2}$$ → eqn. of hyperbola.