Meeting No. 6

I Electrodynamics

1. Maxwell's equations in Gaussian units:

$$\vec{\nabla} \cdot \vec{E}(\vec{x},t) = 4\pi \rho(\vec{x},t)$$

$$\vec{\nabla} \times \vec{E}(\vec{x},t) = -\frac{1}{c} \frac{\partial \vec{B}(\vec{x},t)}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{x},t) = 0$$

$$\vec{\nabla} \times \vec{B}(\vec{x},t) = \frac{1}{c} \frac{\partial \vec{E}(\vec{x},t)}{\partial t} + \frac{4\pi}{c} \vec{J}(\vec{x},t)$$

Lorentz force on a point charge

$$\vec{F}(\vec{x}_a,t) = q\, \vec{E}(\vec{x}_a,t) + q \frac{\vec{V}_a}{c} \times \vec{B}(\vec{x}_a,t)$$

2. Field strength tensor:

$$F^{\mu\nu}(\vec{x},t) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & -B_z & B_y \\ +E_y & +B_z & 0 & -B_x \\ +E_z & -B_y & +B_x & 0 \end{bmatrix}$$

3. 4-current vector:

$$J^\mu(\vec{x},t) = \left(c\rho(\vec{x},t),\ \vec{J}(\vec{x},t)\right)$$

4 (i) Evaluate $F_{\alpha\beta} = g_{\alpha\mu} F^{\mu\nu} g_{\nu\beta}$

(ii) Evaluate $\tilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}$

(iii) Evaluate $\tilde{F}_{\alpha\beta} = g_{\alpha\mu} \tilde{F}^{\mu\nu} g_{\nu\beta}$

5 (i) Show that

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

reproduces the four Maxwell's equations.

(<u>Hint</u> : $\rightarrow$ remember $x^\mu \partial_\mu = t\frac{\partial}{\partial t} + \vec{x}\cdot\vec{\nabla} \neq t\frac{\partial}{\partial t} - \vec{x}\cdot\vec{\nabla}$

$\rightarrow$ $F^{ij} = -\epsilon^{ijk} B_k$, also $F_{ij} = -\epsilon_{ijk} B^k$)

(ii) Show that

$F^{\mu\alpha} U_\alpha$ corresponds to

$\mu = 0 \quad \longrightarrow \quad \gamma(\vec{E}\cdot\vec{v})$

$\mu = i \quad \longrightarrow \quad c\gamma\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$

IV Electromagnetic force on relativistic particles

① We saw

$$f^\alpha = \frac{dp^\alpha}{d\tau} = \gamma\left(\frac{1}{c}\frac{dU}{dt}, \frac{d\vec{p}}{dt}\right)$$

② Electromagnetic force is defined to be.

$$f^\alpha = \frac{q}{c} F^{\alpha\beta} u_\beta$$

③ $$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} u_\beta$$

$$\gamma\left(\frac{1}{c}\frac{dU}{dt}, \frac{d\vec{p}}{dt}\right) = \frac{q}{c}\left(\gamma(\vec{E}\cdot\vec{v}), c\gamma\left\{\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right\}\right)$$

④ Thus the equations of motion are.

$$\frac{dU}{dt} = q(\vec{E}\cdot\vec{v})$$

$$\frac{d\vec{p}}{dt} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$$

III Relativistic particle in a constant magnetic field

① $$\frac{dU}{dt} = q(\vec{E}\cdot\vec{v})$$

$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

② $\vec{E} = 0$ $$\frac{dU}{dt} = 0 \quad \text{— (i)}$$

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\,\vec{v}\times\vec{B} \quad \text{— (ii)}$$

③ $$\frac{dU}{dt} = 0 \Rightarrow \frac{d}{dt}(mc^2\gamma) = 0$$

$$\Rightarrow \gamma = \text{const}$$

④ Using ③ in ② - (ii)

$$\frac{d}{dt}(m\vec{v}\gamma) = \frac{q}{c}\,\vec{v}\times\vec{B}$$

$$\frac{d\vec{v}}{dt} = \frac{q}{mc\gamma}\,\vec{v}\times\vec{B}$$

⑤ Breakdown the vector with respect to $\vec{B}$.

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$$

$$\vec{v}_{\perp}\cdot\vec{B} = 0$$

(6) $$\frac{d\vec{V}}{dt} = \frac{q}{mc\gamma} \vec{V} \times \vec{B}$$

$$\frac{d\vec{V}_{\parallel}}{dt} = 0 \quad \& \quad \frac{d\vec{V}_{\perp}}{dt} = \frac{q}{mc\gamma} \vec{V}_{\perp} \times \vec{B}$$

(7) $$\vec{V}_{\perp} = V_x \hat{i} + V_y \hat{j}$$

$$\frac{dV_x}{dt} = \frac{qB}{mc\gamma} V_y$$

$$\frac{dV_y}{dt} = -\frac{qB}{mc\gamma} V_x$$

(8) Using (6) & (7)

$$V_{\parallel} = \text{const}$$

$$\frac{d^2 V_x}{dt^2} = -\omega_B^2 V_x \qquad \omega_B = \frac{qB}{mc\gamma} \equiv \text{cyclotron frequency}$$

$$\frac{d^2 V_y}{dt^2} = -\omega_B^2 V_y$$

(9) $$V_{\parallel} = \text{const}$$

$$V_x = V_{xo} \sin(\omega_B t + \delta_x)$$

$$V_y = V_{yo} \cos(\omega_B t + \delta_y)$$

(10) $$x - x_o = \frac{V_{xo}}{\omega_B} \cos(\omega_B t + \delta_x)$$

$$y - y_o = -\frac{V_{yo}}{\omega_B} \sin(\omega_B t + \delta_y)$$

$\Bigg] \longrightarrow$ circular motion with radius $R^2 = \dfrac{V_{xo}^2 + V_{yo}^2}{\omega_B^2}$.

$$z - z_o = (\text{const})\, t$$