Solutions

Exam No. 02 (Fall 2013)

PHYS 530B: Quantum Mechanics II

Date: 2013 Oct 31

1. (**20 points.**) A composite system is built out of two angular momenta $j_1 = 7, j_2 = \frac{3}{2}$. Determine the total number of angular momentum states for the composite system.

2. (**20 points.**) Using $\sigma_x\sigma_y = -\sigma_y\sigma_x = i\sigma_z$, evaluate

$$e^{-i\frac{\theta}{2}\sigma_x}\sigma_y e^{i\frac{\theta}{2}\sigma_x}. \tag{1}$$

Express the answer in terms of σ matrices.

3. (**30 points.**) The transformation function relating the angular momentum eigenvectors between two coordinate frames, related by rotations described using Euler angles (ψ, θ, ϕ), is

$$\langle j, m|U(\psi, \theta, \phi)|j', m'\rangle = \delta_{jj'}e^{im\psi}U^{(j)}_{m,m'}(\theta)e^{im'\phi}, \tag{2}$$

where $U^{(j)}_{m,m'}(\theta)$ are generated by the relation

$$\frac{\bar{y}_+^{j+m}}{\sqrt{(j+m)!}}\frac{\bar{y}_-^{j-m}}{\sqrt{(j-m)!}} = \sum_{m'=-j}^{j} e^{im\psi}U^{(j)}_{m,m'}(\theta)e^{im'\phi}\frac{y_+^{j+m'}}{\sqrt{(j+m')!}}\frac{y_-^{j-m'}}{\sqrt{(j-m')!}}, \tag{3}$$

where

$$\begin{bmatrix} \bar{y}_+ \\ \bar{y}_- \end{bmatrix} = \begin{bmatrix} e^{i\frac{\psi}{2}}\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & e^{i\frac{\psi}{2}}\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -e^{-i\frac{\psi}{2}}\sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & e^{-i\frac{\psi}{2}}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{bmatrix}\begin{bmatrix} y_+ \\ y_- \end{bmatrix}. \tag{4}$$

The above transformation function gives the probability amplitude relating measurements of angular momentum, or magnetic dipole moment, in two different directions related by the Euler angles.

The following experiment is performed using beams consisting of angular momentum $j = 2$. An initial Stern-Gerlach measurement selects the $j = 2$, $m = 2$, beam. A second measurement of $j = 2$, $m' = 1$, is made on this beam in a direction differing by angle θ. The probability relating these measurements is determined by

$$p(m, m'; \theta) = |\langle j, m|U(\psi, \theta, \phi)|j', m'\rangle|^2 . \tag{5}$$

Extract the probability, $p(m = 2, m' = 1; \theta)$, for $j = 2$.

4. (**30 points.**) Construct the total angular momentum state $|3, 3\rangle$ for the composite system built out of two angular momenta $j_1 = 3, j_2 = 1$.

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Prob. 1 , Exam-2

$$N = (2j_1+1)(2j_2+1)$$
$$= (2\times 7+1)(2\tfrac{3}{2}+1)$$
$$= 15 \times 4$$
$$= 60$$

Prob 2, Exam-2

$$\sigma_x \sigma_y = -\sigma_y \sigma_x = i\sigma_z$$

$$e^{-i\frac{\theta}{2}\sigma_x}\,\sigma_y\, e^{i\frac{\theta}{2}\sigma_x} = \left(\cos\tfrac{\theta}{2} - i\sigma_x \sin\tfrac{\theta}{2}\right)\sigma_y\, e^{i\frac{\theta}{2}\sigma_x}$$
$$= \sigma_y\left(\cos\tfrac{\theta}{2} + i\sigma_x \sin\tfrac{\theta}{2}\right) e^{i\frac{\theta}{2}\sigma_x}$$
$$= \sigma_y\, e^{i\frac{\theta}{2}\sigma_x}\, e^{i\frac{\theta}{2}\sigma_x}$$
$$= \sigma_y\, e^{i\theta\sigma_x}$$
$$= \sigma_y\left(\cos\theta + i\sigma_x \sin\theta\right)$$
$$= \sigma_y \cos\theta + i\sigma_y\sigma_x \sin\theta$$
$$= \sigma_y \cos\theta + \sigma_z \sin\theta$$

Prob 3, Exam-2

$j = 2$ and $m = 2$

$$\frac{1}{\sqrt{(j+m)!}}\frac{1}{\sqrt{(j-m)!}}\left(e^{i\frac{\psi}{2}}\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + e^{i\frac{\psi}{2}}\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)^{j+m}$$

$$\times\left(-e^{-i\frac{\psi}{2}}\sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + e^{-i\frac{\psi}{2}}\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)^{j-m}$$

$$= \sum_{m'=-2}^{+2} e^{im\psi}\, U^{(j)}_{mm'}(\theta)\, e^{im'\phi}\, \frac{y_+^{\,j+m'}}{\sqrt{(j+m')!}}\, \frac{y_-^{\,j-m'}}{\sqrt{(j-m')!}}$$

$$\frac{1}{\sqrt{4!}}\frac{1}{\sqrt{0!}}\left(e^{i\frac{\psi}{2}}\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + e^{i\frac{\psi}{2}}\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)^{2+2}$$

$$= \sum_{m'=-2}^{2} e^{i2\psi}\, U^{(2)}_{2,m'}(\theta)\, e^{im'\phi}\, \frac{y_+^{\,2+m'}}{\sqrt{(2+m')!}}\, \frac{y_-^{\,2-m'}}{\sqrt{(2-m')!}}$$

Collecting coefficients of. $y_+^3\, y_-^1$ on both sides. we have.

$$\frac{1}{\sqrt{4!}}\frac{1}{\sqrt{0!}}\; 4\left(e^{i\frac{\psi}{2}}\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\right)^3\left(e^{i\frac{\psi}{2}}\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\right)$$

$$= e^{i2\psi}\, U^{(2)}_{2,1}(\theta)\, e^{i\phi}\, \frac{1}{\sqrt{3!}}\,\frac{1}{\sqrt{1!}}$$

$$U^{(2)}_{2,1}(\theta) = \frac{1}{\sqrt{4}}\, 4\cos^3\frac{\theta}{2}\sin\frac{\theta}{2} = \sqrt{4}\cos^3\frac{\theta}{2}\sin\frac{\theta}{2}.$$

$$P(m=2, m=1;\theta) = \left|U^{(2)}_{2,1}(\theta)\right|^2 = 4\cos^6\frac{\theta}{2}\sin^2\frac{\theta}{2}.$$

Prob 4 , Exam - 2

$j_1 = 3 \quad j_2 = 1 \qquad j = 4, 3, 2.$

$$|4,4\rangle = |3,3\rangle_{①} \, |1,1\rangle_{②}$$

$$|4,3\rangle = \sqrt{\tfrac{3}{4}} \, |3,2\rangle_{①} \, |1,1\rangle_{②} + \sqrt{\tfrac{1}{4}} \, |3,3\rangle_{①} \, |1,0\rangle_{②}$$

Let $\quad |3,3\rangle = a \, |3,2\rangle_{①} \, |1,1\rangle_{②} + b \, |3,3\rangle_{①} \, |1,0\rangle_{②}$

$$a^2 + b^2 = 1$$

$$a\sqrt{\tfrac{3}{4}} + b\sqrt{\tfrac{1}{4}} = 0 \quad \Rightarrow \quad b = -\sqrt{3}\, a.$$

$$a^2 + 3a^2 = 1$$

$$a = \sqrt{\tfrac{1}{4}}$$

$$b = -\sqrt{\tfrac{3}{4}}$$

Thus,

$$|3,3\rangle = \sqrt{\tfrac{1}{4}} \, |3,2\rangle_{①} \, |1,1\rangle_{②} - \sqrt{\tfrac{3}{4}} \, |3,3\rangle_{①} \, |1,0\rangle_{②}$$