Solutions

Final Exam (Fall 2013)

PHYS 530B: Quantum Mechanics II

Date: 2013 Dec 9

1. **(25 points.)** Construct the total angular momentum state $|N, N\rangle$ for the composite system built out of two angular momenta $j_1 = N$ and $j_2 = 1$.

2. **(25 points.)** The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Evaluate

$$r^2\,\mathbf{p} - \mathbf{p}\,r^2. \tag{1}$$

3. **(25 points.)** Verify, by substitution, that

$$G(\mathbf{r}) = -\frac{e^{ikr}}{4\pi r} \tag{2}$$

is a particular solution to the Green's function equation

$$\left[\nabla^2 + k^2\right] G(\mathbf{r}) = \delta^{(3)}(\mathbf{r}). \tag{3}$$

4. **(25 points.)** The Dirac equation is described by the Hamiltonian

$$H = \boldsymbol{\alpha} c \cdot \mathbf{p} + \beta m c^2, \tag{4}$$

where $\boldsymbol{\alpha}$ and β are anticommuting operators that satisfy

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \qquad \alpha_i\beta + \beta\alpha_i = 0, \qquad \beta^2 = 1. \tag{5}$$

Evaluate the commutation relation $[\mathbf{r}, H]$. Thus, derive the equation of motion

$$\frac{d\mathbf{r}}{dt} = \frac{1}{i\hbar}\,[\mathbf{r}, H] = ?. \tag{6}$$

Comparing the above relativistic equation of motion with the corresponding non-relativistic equation of motion give a plausible physical interpretation for the operator $\boldsymbol{\alpha} c$.

Prob 1, Final Exam

$J_- |j, m\rangle = \sqrt{(j+m)(j-m+1)}\, |j, m-1\rangle$

$j_1 = N \qquad j_2 = 1$

$j_1 + j_2 = N+1 \qquad j_1 - j_2 = N-1$

$j = N+1,\ N,\ N-1$

$$|N+1, N+1\rangle = |N, N\rangle_{1} |1, 1\rangle_{2}$$

$$\sqrt{2(N+1)}\ |N+1, N\rangle = \sqrt{2N}\ |N, N-1\rangle_{1} |1,1\rangle_{2} + \sqrt{2}\ |N, N\rangle_{1} |1, 0\rangle_{2}$$

$$|N+1, N\rangle = \sqrt{\frac{N}{N+1}}\ |N, N-1\rangle_{1} |1,1\rangle_{2} + \sqrt{\frac{1}{N+1}}\ |N, N\rangle_{1} |1, 0\rangle_{2}$$

$$|N, N\rangle = a\ |N, N-1\rangle_{1} |1,1\rangle_{2} + b\ |N, N\rangle_{1} |1, 0\rangle_{2}$$

$$a^2 + b^2 = 1$$

$$a\sqrt{\frac{N}{N+1}} + b\sqrt{\frac{1}{N+1}} = 0 \Rightarrow b = -a\sqrt{N}$$

$$a^2 + a^2 N = 1 \Rightarrow a = \sqrt{\frac{1}{N+1}} \qquad b = -\sqrt{\frac{N}{N+1}}$$

Thus,

$$|N, N\rangle = \sqrt{\frac{1}{N+1}}\ |N, N-1\rangle_{1} |1,1\rangle_{2} - \sqrt{\frac{N}{N+1}}\ |N, N\rangle_{1} |1, 0\rangle_{2}$$

Prob 2, Final exam

$$r^2 \vec{P}_i - \vec{P}_i r^2 = r_j r_j p_i - p_i r^2$$
$$= r_j (p_i r_j + i\hbar \delta_{ij}) - p_i r^2$$
$$= r_j p_i r_j + i\hbar r_i - p_i r^2$$
$$= (p_i r_j + i\hbar \delta_{ij}) r_j + i\hbar r_i - p_i r^2$$
$$= p_i r^2 + 2i\hbar r_i - p_i r^2$$
$$= 2i\hbar \vec{r}_i$$

Prob 3, Final exam

$\vec{\nabla} r = \hat{r}$

$-\nabla^2 \frac{1}{r} = \delta^{(3)}(\vec{r})$

$\vec{\nabla} \cdot \vec{r} = 3$

$$G(r) = -\frac{e^{ikr}}{4\pi r}$$
$$\vec{\nabla} G = -\left(\vec{\nabla} e^{ikr}\right) \frac{1}{4\pi r} - \frac{e^{ikr}}{4\pi} \vec{\nabla} \frac{1}{r}$$
$$= -ik (\vec{\nabla} r)\, e^{ikr} \frac{1}{4\pi r} - \frac{e^{ikr}}{4\pi} \vec{\nabla} \frac{1}{r}$$
$$= -\frac{ik}{4\pi} \frac{\vec{r}}{r^2} e^{ikr} - \frac{e^{ikr}}{4\pi} \vec{\nabla} \frac{1}{r}$$
$$\nabla^2 G = -\frac{ik}{4\pi} \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2} e^{ikr}\right) - \left(\vec{\nabla} \frac{e^{ikr}}{4\pi}\right) \cdot \vec{\nabla} \frac{1}{r} - \frac{e^{ikr}}{4\pi} \nabla^2 \frac{1}{r}$$
$$= -\frac{ik}{4\pi} (\vec{\nabla} \cdot \vec{r}) \frac{1}{r^2} e^{ikr} - \frac{ik}{4\pi} \vec{r} \cdot \left(\vec{\nabla} \frac{1}{r^2}\right) e^{ikr} - \frac{ik}{4\pi} \frac{\vec{r}}{r^2} \cdot \vec{\nabla} e^{ikr}$$
$$+ \frac{1}{4\pi} \left(\vec{\nabla} e^{ikr}\right) \cdot \frac{\vec{r}}{r^3} + e^{ikr} \delta^{(3)}(\vec{r})$$

$$\nabla^2 G = -\frac{3ik}{4\pi}\frac{1}{r^2}e^{ikr} - \frac{ik}{4\pi}\vec{r}\cdot\frac{(-2)}{r^3}\hat{r}\,e^{ikr} - \frac{ik}{4\pi}\frac{\vec{r}}{r^2}\cdot(ik\hat{r})\,e^{ikr}$$

$$+ \frac{1}{4\pi}(ik\hat{r})\,e^{ikr}\cdot\frac{\vec{r}}{r^3} + \delta^{(3)}(\vec{r})$$

$$= -\frac{3ik}{4\pi}\frac{1}{r^2}e^{ikr} + \frac{2ik}{4\pi}\frac{1}{r^2}e^{ikr} + k^2\frac{e^{ikr}}{4\pi r}$$

$$+ \frac{ik}{4\pi}\frac{1}{r^2}e^{ikr} + \delta^{(3)}(\vec{r})$$

$$= -k^2 G + \delta^{(3)}(\vec{r})$$

✓

$$\Rightarrow \quad (\nabla^2 + k^2)\,G(r) = \delta^{(3)}(\vec{r})$$

Prob 4, Final exam

$$H = \vec{\alpha}c\cdot\vec{p} + \beta mc^2.$$

$$[\vec{r}, H] = [\vec{r}, \vec{\alpha}c\cdot\vec{p}] + \underbrace{[\vec{r}, \beta mc^2]}_{=0}$$

$$= i\hbar\,\vec{\alpha}c.$$

$$\Rightarrow \quad \frac{d\vec{r}}{dt} = \frac{1}{i\hbar}[\vec{r}, H] = \vec{\alpha}c$$

Comparing with $\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$ we can interpret $\vec{\alpha}c$ with the velocity operator.