I Time evolution of dynamic variables

① Consider the time translation

$$t' = t - \delta t$$

generated by the unitary transformation

$$U = 1 + \frac{i}{\hbar} G, \qquad G = -H\delta t.$$

② Let $v(t)$ be a dynamic variable (eg: $\vec{P}$, $\vec{J}$, but not $\vec{N}$)

$$v'(t') = v(t)$$

$$v'(t-\delta t) = v(t) \qquad (t \to t+\delta t)$$

$$v'(t) = v(t+\delta t)$$

$$v(t) - \delta v = v(t) + \delta t \frac{d}{dt} v$$

$$\Rightarrow \quad \delta v = -\delta t \frac{d}{dt} v.$$

③ Comparing this with

$$\delta v = \frac{1}{i\hbar}[v, -H\delta t]$$

we have.

$$-\cancel{\delta t} \frac{d}{dt} v = \frac{1}{i\hbar}[v, -H\cancel{\delta t}]$$

$$\frac{dv}{dt} = \frac{1}{i\hbar}[v, H].$$

(4) If we have an operator constructed out of dynamic variables and explicit time dependence (eg: $\vec{N} = \vec{P}t - M\vec{R}$)

$$F(v(t), t)$$

then

$$\begin{aligned} F' &= F(U^{-1} v(t) U, t) \\ &= F(v'(t), t) \\ &= F(v(t), t) - \delta F \\ &= F(v(t), t) - \frac{1}{i\hbar}[F, -H\delta t] \end{aligned}$$

(5)

$$F(v'(t), t) = F(v(t), t) - \frac{1}{i\hbar}[F, -H\delta t]$$

$v'(t) = v(t+\delta t)$

$$\frac{F(v(t+\delta t), t) - F(v(t), t)}{\delta t} = \frac{1}{i\hbar}[F, H]$$

$$\frac{dF}{dt} - \frac{\partial F}{\partial t} = \frac{1}{i\hbar}[F, H]$$

(6) Thus, the time evolution of dynamic varible is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i\hbar}[F, H].$$

7 **Example 1**

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{1}{i\hbar}\underbrace{[H,H]}_{=0}$$

$$= \frac{\partial H}{\partial t}$$

Thus, if H has no explicit time dependence,

$$\frac{dH}{dt} = 0,$$

in which case H = energy is conserved.

8 **Example 2**

$$\frac{d\vec{P}}{dt} = \frac{\partial \vec{P}}{\partial t} + \frac{1}{i\hbar}\underbrace{[\vec{P},H]}_{=0}$$

Thus, momentum is conserved. Momentum operator has no explicit time dependence.

$$\frac{d\vec{P}}{dt} = \frac{1}{i\hbar}[\vec{P},H] = 0$$

9 **Example 3**

Since position operator has explicit time dependence we have

$$\frac{d\vec{R}}{dt} = \frac{1}{i\hbar}[\vec{R},H] = \frac{1}{i\hbar}\left[\vec{P}\frac{t}{M} - \frac{\vec{N}}{M}, H\right]$$

$$= \frac{\vec{P}}{M}.$$

II Hamilton equations of motion

(10) Recollect Hamilton equations of motion in classical mechanics.

$$\left.\begin{aligned} \frac{d\vec{P}}{dt} &= -\frac{\partial H}{\partial \vec{R}} \\ \frac{d\vec{R}}{dt} &= \frac{\partial H}{\partial \vec{P}} \end{aligned}\right\} \rightarrow \text{Newtonian}$$

(11) Do they hold in quantum mechanics? In (8) & (9) we have seen

$$\left.\begin{aligned} \frac{d\vec{P}}{dt} &= \frac{1}{i\hbar}[\vec{P}, H] = 0 \\ \frac{d\vec{R}}{dt} &= \frac{1}{i\hbar}[\vec{R}, H] = \frac{\vec{P}}{M} . \end{aligned}\right\} \rightarrow \text{Quantum.}$$

(12) Let us consider

$$K(a) = e^{-\frac{i}{\hbar}ap} \, x \, e^{\frac{i}{\hbar}ap}$$

where 'a' is a number and $[x, p] = i\hbar$.

Differentiating we have.

$$\frac{\partial K}{\partial a} = e^{-\frac{i}{\hbar}ap}\left[-\frac{i}{\hbar}px + \frac{i}{\hbar}xp\right] e^{\frac{i}{\hbar}ap}$$

$$= e^{-\frac{i}{\hbar}ap}(-1)\, e^{\frac{i}{\hbar}ap}$$

$$= -1$$

(13) $$\frac{\partial K}{\partial a} = -1$$

$$\Rightarrow \quad K(a) = -a + \text{const}$$

where const is independent of a. At $a=0$ we know that $K(0) = x$, which implies.

$$\text{const} = x$$

Thus,

$$K(a) = x - a ,$$

and we have proved

$$e^{-\frac{i}{\hbar} a p} \, x \, e^{\frac{i}{\hbar} a p} = x - a .$$

(14) Similarly we can prove.

(i) $$e^{-\frac{i}{\hbar} \vec{R}' \cdot \vec{P}} \, \vec{R} \, e^{\frac{i}{\hbar} \vec{R}' \cdot \vec{P}} = \vec{R} - \vec{R}'$$ ($\vec{R}'$ is numerical.)

(ii) $$e^{\frac{i}{\hbar} \vec{R} \cdot \vec{P}'} \, \vec{P} \, e^{-\frac{i}{\hbar} \vec{R} \cdot \vec{P}'} = \vec{P} - \vec{P}'$$ ($\vec{P}'$ is numerical.)

(15) And, more generally.

(i) $$e^{-\frac{i}{\hbar} \vec{R}' \cdot \vec{P}} \, F(\vec{R}, \vec{P}) \, e^{\frac{i}{\hbar} \vec{R}' \cdot \vec{P}} = F(\vec{R} - \vec{R}', \vec{P})$$

(ii) $$e^{\frac{i}{\hbar} \vec{R} \cdot \vec{P}'} \, F(\vec{R}, \vec{P}) \, e^{-\frac{i}{\hbar} \vec{R} \cdot \vec{P}'} = F(\vec{R}, \vec{P} - \vec{P}')$$

16 For infinitesimal $\vec{R}'$ and $\vec{P}'$ we have.

(i)
$$e^{-\frac{i}{\hbar}\delta\vec{R}'\cdot\vec{P}} F(\vec{R},\vec{P})\, e^{\frac{i}{\hbar}\delta\vec{R}'\cdot\vec{P}} = F(\vec{R}-\delta\vec{R}',\vec{P})$$
$$= F(\vec{R},\vec{P}) - \delta\vec{R}'\cdot\frac{\partial}{\partial\vec{R}}F$$

$$\Rightarrow \quad \frac{1}{i\hbar}\left[F,\ \delta\vec{R}'\cdot\vec{P}\right] = \delta\vec{R}'\cdot\frac{\partial}{\partial\vec{R}}F$$

$$\frac{1}{i\hbar}\left[\vec{P},\ F\right] = -\frac{\partial}{\partial\vec{R}}F$$

(ii)
$$e^{\frac{i}{\hbar}\vec{R}\cdot\delta\vec{P}'} F(\vec{R},\vec{P})\, e^{-\frac{i}{\hbar}\vec{R}\cdot\delta\vec{P}'} = F(\vec{R},\vec{P}-\delta\vec{P}')$$
$$= F(\vec{R},\vec{P}) - \delta\vec{P}'\cdot\frac{\partial}{\partial\vec{P}}F$$

$$\Rightarrow \quad \frac{1}{i\hbar}\left[F,\ -\vec{R}\cdot\delta\vec{P}'\right] = \delta\vec{P}'\cdot\frac{\partial}{\partial\vec{P}}F$$

$$\frac{1}{i\hbar}\left[\vec{R},\ F\right] = \frac{\partial}{\partial\vec{P}}F$$

17 For $F = H$ we have.

$$\frac{1}{i\hbar}\left[\vec{R},H\right] = \frac{\partial}{\partial\vec{P}}H$$

$$\frac{1}{i\hbar}\left[\vec{P},H\right] = -\frac{\partial}{\partial\vec{R}}H$$

18 Using 17 in 11 we have.

$$\frac{d\vec{P}}{dt} = \frac{1}{i\hbar}[\vec{P}, H] = -\frac{\partial H}{\partial \vec{R}} = 0$$

$$\frac{d\vec{R}}{dt} = \frac{1}{i\hbar}[\vec{R}, H] = \frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}}{M}.$$

Thus, the Hamilton equations are true in quantum mechanics.

19 We can have some informatn about the structure of H from 18.

$$\frac{\partial H}{\partial \vec{R}} = 0 \Rightarrow H \text{ is independent of } \vec{R}.$$

$$\frac{\partial H}{\partial \vec{P}} = \frac{\vec{P}}{M} \Rightarrow H = \frac{\vec{P}^2}{2M} + H_{internal} \qquad \frac{\partial H_{internal}}{\partial \vec{P}} = 0.$$