SET-07

Construction of basis set for angular momentum

(1) The commutation relations for the angular momentum generators are

$$\frac{1}{i\hbar}\left[J_i , J_j\right] = \epsilon_{ijk} J_k ,$$

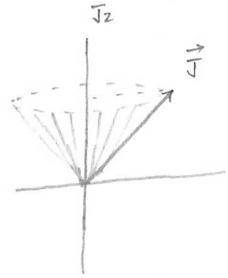
which is a statement of transformation under rotations,

$$\frac{1}{i\hbar}\left[\vec{J}, \delta\vec{\omega}\cdot\vec{J}\right] = \delta\vec{\omega}\times\vec{J}.$$

(2) Since $J^2 = \vec{J}\cdot\vec{J}$ is scalar we have

$$\frac{1}{i\hbar}\left[J^2, \vec{J}\right] = 0$$

(3) Thus, J_z, for example, can have simultaneous eigen functions with J^2. Since $[J_x , J_z] \neq 0$ and $[J_y , J_z] \neq 0$, we can not have further simultaneous measurements. That is, measurements of J_x and J_y will be uncertain if J^2 and J_z has been specified.



→ Since J_x and J_y are uncertain we can only specify the angular momentum vector on a cone with specific magnitude $|\vec{J}|$.

(4) Define dimensionless generators

$$\tilde{J}_i = \frac{1}{\hbar} J_i .$$

Thus.

$$\frac{1}{i} [J_i , J_j] = \epsilon_{ijk} J_k .$$

(5) Let the common eigen states of J^2 and J_z be $|j, m\rangle$, where

$$\tilde{J}^2 \, | j, m\rangle = j(j+1) \, |j, m\rangle ,$$

$$J_z \, | j, m\rangle = m \, | j, m\rangle .$$

(6) The eigen states $|j, m\rangle$ can be chosen to be orthonormal (see HW-02). Thus,

$$\langle j, m | j', m'\rangle = \delta_{jj'} \, \delta_{mm'} .$$

(7) Let us construct the operators (ladder operators)

$$\tilde{J}_+ = \tilde{J}_x + i \, \tilde{J}_y ,$$

$$\tilde{J}_- = \tilde{J}_x - i \, \tilde{J}_y ,$$

which are non-hermitian, but related by

$$\tilde{J}_+^{\dagger} = \tilde{J}_- .$$

8 Using commutation relations for $\vec{\tilde{J}}$ we have.

$$[\tilde{J}_z, \tilde{J}_+] = \tilde{J}_z(\tilde{J}_x + i\tilde{J}_y) - (\tilde{J}_x + i\tilde{J}_y)\tilde{J}_z$$
$$= [\tilde{J}_z, \tilde{J}_x] + i[\tilde{J}_z, \tilde{J}_y]$$
$$= i\tilde{J}_y + \tilde{J}_x$$
$$= \tilde{J}_+$$

$$[\tilde{J}_z, \tilde{J}_-] = [\tilde{J}_z, \tilde{J}_+^\dagger] \qquad (\text{using } [A,B]^\dagger = [B^\dagger, A^\dagger]$$
$$= [\tilde{J}_+, \tilde{J}_z]^\dagger$$
$$= -\tilde{J}_+^\dagger$$
$$= -\tilde{J}_-$$

9 $$\tilde{J}_+\tilde{J}_- = (\tilde{J}_x + i\tilde{J}_y)(\tilde{J}_x - i\tilde{J}_y)$$
$$= \tilde{J}_x^2 + \tilde{J}_y^2 - i[\tilde{J}_x, \tilde{J}_y]$$
$$= \tilde{J}_x^2 + \tilde{J}_y^2 + \tilde{J}_z$$
$$= \tilde{J}^2 - \tilde{J}_z^2 + \tilde{J}_z$$

$$\tilde{J}_-\tilde{J}_+ = (\tilde{J}_x - i\tilde{J}_y)(\tilde{J}_x + i\tilde{J}_y)$$
$$= \tilde{J}_x^2 + \tilde{J}_y^2 + i[\tilde{J}_x, \tilde{J}_y]$$
$$= \tilde{J}_x^2 + \tilde{J}_y^2 - \tilde{J}_z$$
$$= \tilde{J}^2 - \tilde{J}_z^2 - \tilde{J}_z$$

$$[\tilde{J}_+, \tilde{J}_-] = 2\tilde{J}_z$$

10 Using 8 and 9 we have.

$$\tilde{J}_z \tilde{J}_+ |j,m\rangle = (m+1)\, \tilde{J}_+ |j,m\rangle \qquad \text{— (i)}$$

$$\tilde{J}_z \tilde{J}_- |j,m\rangle = (m-1)\, \tilde{J}_- |j,m\rangle \qquad \text{— (ii)}$$

$$\tilde{J}_- \tilde{J}_+ |j,m\rangle = \{ j(j+1) - m(m+1) \} |j,m\rangle \qquad \text{— (iii)}$$

$$\tilde{J}_+ \tilde{J}_- |j,m\rangle = \{ j(j+1) - m(m-1) \} |j,m\rangle \qquad \text{— (iv)}$$

11 For any non-hermitian operator A, the operator construction $A^\dagger A$ is hermitian. Thus, $A^\dagger A$ will have non-negative eigen values. Since

$$\tilde{J}_- \tilde{J}_+ = \tilde{J}_+^\dagger \tilde{J}_+$$

conclude using 10 – (iii) that

$$j(j+1) - m(m+1) = (j-m)(j+m+1) \geq 0$$

and conclude using 10 (iv) that

$$j(j+1) - m(m-1) = (j+m)(j-m+1) \geq 0$$

12 Using 10 –(i) and $[\tilde{J}^2, \tilde{J}_+] = 0$ conclude

$$\tilde{J}_+ |j,m\rangle = A_{jm} |j,m+1\rangle$$

and using 10 –(ii) and $[\tilde{J}^2, \tilde{J}_-] = 0$ conclude.

$$\tilde{J}_- |j,m\rangle = B_{jm} |j,m-1\rangle .$$

(13) Since $j \geq 0$ and we require to satisfy (11) there exists a maximum m, $\bar{m}$, that satisfies $\tilde{J}_+ | j, \bar{m} \rangle = 0$

$$(j-\bar{m})(j+\bar{m}+1) = 0$$

which implies

$$\bar{m} = j$$

because $\bar{m} = -j-1$ does not satisfy $j \geq 0$.

(14) Similarly, there exists a minimum m, $\underline{m}$, satisfying

$$(j+\underline{m})(j-\underline{m}+1) = 0$$

which implies

$$\underline{m} = -j$$

because $\underline{m} = j+1$ does not satisfy $j \geq 0$.

(15) The descent $\bar{m} = j$ to $\underline{m} = -j$ occurs is

$$n = 0, 1, 2, \ldots$$

steps. Therefore

$$2j = n,$$

or

$$j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots$$

16 Using 10 -(iii)

$$\langle j m | \tilde{J}_- \tilde{J}_+ | j m \rangle = j(j+1) - m(m+1)$$

$$\left(\tilde{J}_- | j m \rangle \right)^\dagger \tilde{J}_+ | j m \rangle = j(j+1) - m(m+1)$$

$$\left(A_{jm}^* | j m \rangle^\dagger \right) A_{jm} | j m \rangle = j(j+1) - m(m+1)$$

$$\Rightarrow \quad |A_{jm}|^2 = j(j+1) - m(m+1)$$
$$= (j-m)(j+m+1)$$

17 Similarly, using 10 -(iv) we have.

$$|B_{jm}|^2 = j(j+1) - m(m-1)$$
$$= (j+m)(j-m+1)$$

18 Using 10 -(iii) we also have.

$$\tilde{J}_- \tilde{J}_+ | j m \rangle = A_{jm} \tilde{J}_- | j, m+1 \rangle$$
$$= A_{jm} B_{j,m+1} | j m \rangle$$

which implies.

$$A_{jm} B_{j,m+1} = j(j+1) - m(m+1) = |A_{jm}|^*$$

$$\Rightarrow \quad B_{j,m+1} = A_{jm}^* .$$

(19) Summary:

$$\frac{1}{i\hbar}[J_i, J_j] = \epsilon_{ijk} J_k$$

$$\frac{1}{i\hbar}[J^2, J_i] = 0$$

$$J^2 |j,m\rangle = \hbar^2 j(j+1) |j,m\rangle$$

$j = \frac{n}{2}$ $n = 0, 1, 2, \dots$

$m = j, j-1, \dots, -j$.

$$J_z |j,m\rangle = \hbar m |j,m\rangle$$

$$J_+ |j,m\rangle = \hbar \sqrt{(j-m)(j+m+1)} \, |j, m+1\rangle$$

$$J_- |j,m\rangle = \hbar \sqrt{(j+m)(j-m+1)} \, |j, m-1\rangle$$

(20) Matrix representation for fixed j.

$$\underset{\text{matrix}}{J_z} = \begin{bmatrix} \langle j,j | J_z | j,j\rangle & \langle j,j | J_z | j,j-1\rangle & \cdots & \langle j,j | J_z | j,-j\rangle \\ \langle j,j-1 | J_z | j,j\rangle & \langle j,j-1 | J_z | j,j-1\rangle & \cdots & \langle j,j-1 | J_z | j,-j\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle j,-j | J_z | j,j\rangle & \langle j,-j | J_z | j,j-1\rangle & \cdots & \langle j,-j | J_z | j,-j\rangle \end{bmatrix}$$

21 Let $j = \frac{1}{2}$.

$$J_z = \begin{bmatrix} \langle \frac{1}{2}, \frac{1}{2} | J_z | \frac{1}{2}, \frac{1}{2} \rangle & \langle \frac{1}{2}, \frac{1}{2} | J_z | \frac{1}{2}, -\frac{1}{2} \rangle \\ \langle \frac{1}{2}, -\frac{1}{2} | J_z | \frac{1}{2}, \frac{1}{2} \rangle & \langle \frac{1}{2}, -\frac{1}{2} | J_z | \frac{1}{2}, -\frac{1}{2} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\hbar}{2} \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle & -\frac{\hbar}{2} \langle \frac{1}{2}, \frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle \\ \frac{\hbar}{2} \langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, \frac{1}{2} \rangle & -\frac{\hbar}{2} \langle \frac{1}{2}, -\frac{1}{2} | \frac{1}{2}, -\frac{1}{2} \rangle \end{bmatrix}$$

$$= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar}{2} \sigma_z$$

$$J_+ = \begin{bmatrix} \langle \frac{1}{2}, \frac{1}{2} | J_+ | \frac{1}{2}, \frac{1}{2} \rangle & \langle \frac{1}{2}, \frac{1}{2} | J_+ | \frac{1}{2}, -\frac{1}{2} \rangle \\ \langle \frac{1}{2}, -\frac{1}{2} | J_+ | \frac{1}{2}, \frac{1}{2} \rangle & \langle \frac{1}{2}, -\frac{1}{2} | J_+ | \frac{1}{2}, -\frac{1}{2} \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \hbar \sqrt{\left(\frac{1}{2} + \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{2} + 1\right)} \\ 0 & 0 \end{bmatrix} = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$J_+ = J_x + i J_y$$
$$J_- = J_x - i J_y$$

$$J_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$J_x = \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_x$$

$$J_y = \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2i} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2} \sigma_y .$$