SET-08

Harmonic oscillator

1. We constructed eigen state for angular momentum:

$$\frac{1}{i\hbar}[J_i, J_j] = \epsilon_{ijk} J_k$$

$$\frac{1}{i\hbar}[J^2, J_i] = 0$$

$$\frac{1}{\hbar^2} J^2 |j,m\rangle = j(j+1)|j,m\rangle$$

$$\frac{1}{\hbar} J_z |j,m\rangle = m|j,m\rangle$$

$$\frac{1}{\hbar} J_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle$$

$$\frac{1}{\hbar} J_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$$

$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$

$m = j, j-1, \dots, -j.$

2. Let us define

$$n_+ = j+m \qquad n_+ = 0, 1, 2, \dots$$

$$n_- = j-m \qquad n_- = 0, 1, 2, \dots$$

such that

$$j = \frac{1}{2}(n_+ + n_-)$$

$$m = \frac{1}{2}(n_+ - n_-)$$

(3) Using (2) in (1)

$$|j, m\rangle = |n_+, n_-\rangle \quad \text{— relabelling the states}$$

$$\frac{1}{\hbar^2} J^2 |n_+, n_-\rangle = \frac{1}{2}(n_+ + n_-)\left\{\frac{1}{2}(n_+ + n_-) + 1\right\} |n_+, n_-\rangle$$

$$= \left\{\frac{1}{4}(n_+ + n_-)^2 + \frac{1}{2}(n_+ + n_-)\right\} |n_+, n_-\rangle$$

$$= \left[\left\{\frac{1}{2}(n_+ + n_-) + \frac{1}{2}\right\}^2 - \frac{1}{4}\right] |n_+, n_-\rangle$$

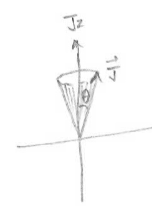
$$\frac{1}{\hbar} J_z |n_+, n_-\rangle = \frac{1}{2}(n_+ - n_-) |n_+, n_-\rangle$$

$$\frac{1}{\hbar} J_+ |n_+, n_-\rangle = \sqrt{n_+ + 1}\,\sqrt{n_-}\,|n_+ + 1, n_- - 1\rangle \quad \rightarrow \quad \text{using } n_+ + 1 = j + (m+1),\; n_- - 1 = j - (m+1)$$

$$\frac{1}{\hbar} J_- |n_+, n_-\rangle = \sqrt{n_+}\,\sqrt{n_- + 1}\,|n_+ - 1, n_- + 1\rangle$$

(4) Aren't there two harmonic oscillators?

For example, if $j \gg 1$ and $m \gg 1$ keeping $j - m = n_-$ finite,



$\theta \approx 0$.

then we have

$$\frac{1}{\hbar} J_+ |2j, n_-\rangle = \sqrt{2j}\,\sqrt{n_-}\,|2j, n_- - 1\rangle$$

5 Consider the Hamiltonian for a harmonic oscillator

$$H = \frac{1}{2} p^2 + \frac{1}{2} \omega^2 x^2 \qquad \omega = 1.$$

where $\frac{1}{i\hbar}[x, p] = 1.$

6 Let us introduce non-hermitian (ladder) operator

$$y = \frac{1}{\sqrt{2\hbar}} (x + ip)$$

$$y^\dagger = \frac{1}{\sqrt{2\hbar}} (x - ip)$$

7
$$\begin{aligned}
[y, y^\dagger] &= \frac{1}{2\hbar} [x + ip, x - ip] \\
&= \frac{1}{2\hbar} \frac{1}{i} [x, p] + \frac{1}{2\hbar} i [p, x] \\
&= \frac{1}{i\hbar} [x, p] \\
&= 1.
\end{aligned}$$

8
$$\begin{aligned}
y^\dagger y &= \frac{1}{2\hbar} (x - ip)(x + ip) \\
&= \frac{1}{2\hbar} \left(x^2 + p^2 + i[x, p] \right) \\
&= \frac{1}{\hbar} H - \frac{1}{2i\hbar} [x, p] \\
&= \frac{1}{\hbar} H - \frac{1}{2}
\end{aligned}$$

$$H = \left(y^\dagger y + \frac{1}{2} \right) \hbar \omega \qquad \omega = 1.$$

9 Note that $y^\dagger y$ is hermitian. Let $|n\rangle$ be the eigen states of $y^\dagger y$.

$$y^\dagger y \, |n\rangle = n \, |n\rangle$$

10 For any operator A, not necessarily hermitian, $A^\dagger A$ is hermitian. Further, $A^\dagger A$ has non-negative eigen values. Thus, Thus $y^\dagger y$ has non-negative eigen value.

$$n \geqslant 0 .$$

11
$$\begin{aligned}
[y^\dagger y, y] &= y^\dagger y y - y y^\dagger y \\
&= y^\dagger y y - (y^\dagger y + 1) y \\
&= - y
\end{aligned}$$

$$\begin{aligned}
[y^\dagger y, y^\dagger] &= y^\dagger y y^\dagger - y^\dagger y^\dagger y \\
&= y^\dagger (y^\dagger y + 1) - y^\dagger y^\dagger y \\
&= y^\dagger
\end{aligned}$$

12 Using 11

$$(y^\dagger y)\, y\, |n\rangle \; - \; y\,(y^\dagger y)\, |n\rangle \; = \; -\, y\, |n\rangle$$

$$(y^\dagger y) \{ y\, |n\rangle \} \; = \; (n-1) \{ y\, |n\rangle \}$$

$$\Rightarrow \quad y\, |n\rangle \; = \; c_n\, |n-1\rangle$$

13 Using 11

$$(y^\dagger y)\, y^\dagger\, |n\rangle \; - \; y^\dagger\,(y^\dagger y)\, |n\rangle \; = \; y^\dagger\, |n\rangle$$

$$(y^\dagger y) \{ y^\dagger\, |n\rangle \} \; = \; (n+1) \{ y^\dagger\, |n\rangle \}$$

$$\Rightarrow \quad y^\dagger\, |n\rangle \; = \; d_n\, |n+1\rangle$$

14 For the compatibility of 10 $(n \geqslant 0)$ and 12 $(y|n\rangle = c_n|n-1\rangle)$ we require the existance of a ground state, such that

$$y\, |0\rangle \; = \; 0\,,$$

which also implies

$$y^\dagger y\, |0\rangle = 0 .$$

15 Using 13 we have

$$y^{\dagger}|0\rangle = d_0|1\rangle$$

$$(y^{\dagger})^2|0\rangle = d_0 d_1|2\rangle$$

$$(y^{\dagger})^n|0\rangle = d_0 d_1 \ldots d_{n-1}|n\rangle \qquad D_n = d_0 d_1 \ldots d_{n-1}$$

Thus, $|n\rangle = \dfrac{(y^{\dagger})^n}{D_n}|0\rangle$

16 $$\langle m|n\rangle = \left\{\frac{(y^{\dagger})^m}{D_m}|0\rangle\right\}^{\dagger} \frac{(y^{\dagger})^n}{D_n}|0\rangle$$

$$= \frac{1}{D_m}\frac{1}{D_n}\langle 0|\, y^m (y^{\dagger})^n |0\rangle$$

17 $$[y,(y^{\dagger})^m] = [y,y^{\dagger}](y^{\dagger})^{m-1} + y^{\dagger}[y,(y^{\dagger})^{m-1}]$$

$$= (y^{\dagger})^{m-1} + y^{\dagger}[y,(y^{\dagger})^{m-1}]$$

$$= (y^{\dagger})^{m-1} + y^{\dagger}\left([y,y^{\dagger}](y^{\dagger})^{m-2} + y^{\dagger}[y,(y^{\dagger})^{m-2}]\right)$$

$$= 2(y^{\dagger})^{m-1} + (y^{\dagger})^2[y,(y^{\dagger})^{m-2}]$$

$$\downarrow$$

$$= m(y^{\dagger})^{m-1}$$

18 $$y^m(y^{\dagger})^n = y^{m-1}\, y\, (y^{\dagger})^n$$

$$= y^{m-1}\left\{(y^{\dagger})^n y + n(y^{\dagger})^{n-1}\right\}$$

$(\because\ y|0\rangle = 0)$

$$y^m(y^{\dagger})^n|0\rangle = n\, y^{m-1}(y^{\dagger})^{n-1}|0\rangle$$

$$= n(n-1)\, y^{m-2}(y^{\dagger})^{n-2}|0\rangle$$

$$\downarrow$$

$$= n!\,|0\rangle\,\delta_{mn} \qquad (\text{for } m \geq n)$$

19 Similarly

$$\langle 0 | y^m (y^\dagger)^n = \langle 0 | \, n! \, \delta_{mn} \qquad \text{for } m \leq n.$$

20 Thus, we have.

$$\langle 0 | y^m (y^\dagger)^n | 0 \rangle = n! \, \delta_{mn}.$$

21 Comparing 16 and 20

$$\langle m | n \rangle = \frac{1}{D_m D_n} n! \, \delta_{mn}$$

$$\Rightarrow \quad D_n = \sqrt{n!}$$

22 Using 21 in 15

$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}} |0\rangle$$

and $D_n = d_0 d_1 \ldots d_{n-1} = \sqrt{n!}$

$$\Rightarrow \quad d_n = \sqrt{n+1}$$

23 Thus, using 13, or using 22

$$|n\rangle = \frac{y^\dagger}{\sqrt{n}} |n-1\rangle$$

we have

$$y^\dagger |n\rangle = \sqrt{n+1} \, |n+1\rangle$$

(24)

$$y^\dagger |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$y\, y^\dagger |n\rangle = \sqrt{n+1}\ y\, |n+1\rangle$$

$$(y^\dagger y + 1)\, |n\rangle = \sqrt{n+1}\ y\, |n+1\rangle$$

$$(n+1)\, |n\rangle = \sqrt{n+1}\ y\, |n+1\rangle$$

$$y\, |n+1\rangle = \sqrt{n+1}\ |n\rangle$$

$$y\, |n\rangle = \sqrt{n}\ |n-1\rangle$$

(25) Thus, we have

$$[y, y^\dagger] = 1$$

$$y^\dagger y\, |n\rangle = n\, |n\rangle \qquad n = 0, 1, 2, \cdots$$

$$y^\dagger |n\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$y\, |n\rangle = \sqrt{n}\ |n-1\rangle$$

(26) Let us find the eigen functions.

$$y\, |0\rangle = 0$$

$$\frac{1}{\sqrt{2\hbar}} (x + ip)\, |0\rangle = 0$$

$$\left(x + \frac{\partial}{\partial x}\right) \psi_0(x) = 0$$

$$\psi_0(x) = \text{const}\ e^{-\frac{x^2}{2}}$$

$$= \frac{1}{\pi^{1/4}}\ e^{-\frac{x^2}{2}}$$

$$\int_{-\infty}^{+\infty} dx\ \psi_0^\dagger(x)\, \psi_0(x) = 1$$

$$(\text{const})^2 \int_{-\infty}^{+\infty} dx\ e^{-x^2} = 1$$

$$(\text{const})^2 = \frac{1}{\sqrt{\pi}}$$

$$\text{const} = \frac{1}{\pi^{1/4}}$$

$$y^\dagger = x - ip = x - \frac{\partial}{\partial x}.$$

(27)
$$|n\rangle = \frac{(y^\dagger)^n}{\sqrt{n!}} |0\rangle$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2\hbar}}\right)^n \left(x - \frac{\partial}{\partial x}\right)^n \psi_0(x)$$

$$= \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2\hbar}}\right)^n \left(x - \frac{\partial}{\partial x}\right)^n e^{-\frac{x^2}{2}}$$

(28)
$$\left(e^{\frac{x^2}{2}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}\right) f = \left(-x + \frac{\partial}{\partial x}\right) f$$

$$= -\left(x - \frac{\partial}{\partial x}\right) f$$

(29)
$$\left(e^{\frac{x^2}{2}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}\right)^2 f = (-1)^2 \left(x - \frac{\partial}{\partial x}\right)^2 f$$

and in general

$$\left(e^{\frac{x^2}{2}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}\right)^n f = (-1)^n \left(x - \frac{\partial}{\partial x}\right)^n f.$$

(30) Further

$$\left(e^{\frac{x^2}{2}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}}\right)^2 f = e^{\frac{x^2}{2}} \frac{\partial}{\partial x} \underbrace{e^{-\frac{x^2}{2}} e^{\frac{x^2}{2}}} \frac{\partial}{\partial x} e^{-\frac{x^2}{2}} f$$

$$= e^{\frac{x^2}{2}} \left(\frac{\partial}{\partial x}\right)^2 e^{-\frac{x^2}{2}} f.$$

(31) Thus,

$$\left(x - \frac{\partial}{\partial x}\right)^n f = (-1)^n e^{\frac{x^2}{2}} \left(\frac{\partial}{\partial x}\right)^n e^{-\frac{x^2}{2}} f.$$

32 Using $f = e^{-\frac{x^2}{2}}$ in 31

$$\left(x - \frac{\partial}{\partial x}\right)^n e^{-\frac{x^2}{2}} = (-1)^n e^{\frac{x^2}{2}} \left(\frac{\partial}{\partial x}\right)^n e^{-x^2}$$

$$= e^{-\frac{x^2}{2}} \underbrace{(-1)^n e^{x^2} \left(\frac{\partial}{\partial x}\right)^n e^{-x^2}}_{H_n(x)}$$

33 Using 32 in 27 we have

$$\psi_n(x) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2\hbar}}\right)^n e^{-\frac{x^2}{2}} H_n(x)$$

34 $H_n(x)$ is a polynomial of order n.

$$H_0(x) = (-1)^0 e^{x^2} e^{-x^2} = 1$$

$$H_1(x) = (-1) e^{x^2} \frac{\partial}{\partial x} e^{-x^2} = 2x$$

$$H_2(x) = (-1)^2 e^{x^2} \frac{\partial}{\partial x}\left(-2x\, e^{-x^2}\right)$$

$$= -2 - 2x(-2x)$$

$$= 4x^2 - 2.$$

(35) Summary :

$$[y, y^\dagger] = 1$$

$$y^\dagger y \, |n\rangle = n \, |n\rangle \qquad n = 0, 1, 2, \ldots$$

$$y^\dagger \, |n\rangle = \sqrt{n+1} \, |n+1\rangle$$

$$y \, |n\rangle = \sqrt{n} \, |n-1\rangle$$

Realization in position basis

$$\psi_0(x) = \frac{1}{\pi^{1/4}} e^{-\frac{x^2}{2}}$$

$$\psi_n(x) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{n!}} \left(\frac{1}{\sqrt{2\hbar}}\right)^n e^{-\frac{x^2}{2}} H_n(x)$$

where

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{\partial}{\partial x}\right)^n e^{-x^2} .$$