Angular momentum decomposition into harmonic oscillators

① Harmonic oscillator :

$$[y, y^\dagger] = 1$$

$n = 0, 1, 2, \dots$

$$y^\dagger y \, |n\rangle = n\,|n\rangle$$

$$y^\dagger \, |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$y \, |n\rangle = \sqrt{n}\,|n-1\rangle$$

② Angular momentum :

$$\frac{1}{i\hbar}[J_i, J_j] = \epsilon_{ijk} J_k$$

$$\frac{1}{\hbar^2} J^2 |j,m\rangle = j(j+1)\,|j,m\rangle$$

$$\frac{1}{\hbar} J_z |j,m\rangle = m\,|j,m\rangle$$

$$\frac{1}{\hbar} J_+ |j,m\rangle = \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle$$

$$\frac{1}{\hbar} J_- |j,m\rangle = \sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$$

$$\frac{1}{\hbar^2} J^2 |n_+, n_-\rangle = \left[\left\{\tfrac{1}{2}(n_+ + n_-) + \tfrac{1}{2}\right\}^2 - \tfrac{1}{4}\right] |n_+, n_-\rangle$$

$$\frac{1}{\hbar} J_z |n_+, n_-\rangle = \tfrac{1}{2}(n_+ - n_-)\,|n_+, n_-\rangle$$

$$\frac{1}{\hbar} J_+ |n_+, n_-\rangle = \sqrt{n_+ + 1}\,\sqrt{n_-}\,|n_+ + 1, n_- - 1\rangle$$

$$\frac{1}{\hbar} J_- |n_+, n_-\rangle = \sqrt{n_+}\,\sqrt{n_- + 1}\,|n_+ - 1, n_- + 1\rangle$$

③ Thus, we construct

$$|n_+, n_-\rangle = |n_+\rangle\,|n_-\rangle$$

$$\frac{1}{\hbar} J_+ = y_+^\dagger\, y_-$$

$$\frac{1}{\hbar} J_- = y_+\, y_-^\dagger$$

$$[y_+, y_+^\dagger] = 1$$

$$[y_-, y_-^\dagger] = 1$$

$$[y_+, y_-] = 0$$

$$[y_+, y_-^\dagger] = 0 .$$

$$\left[\frac{1}{\hbar}J_+ , \frac{1}{\hbar}J_-\right] = 2\frac{1}{\hbar}J_z$$

(4)

$$\frac{1}{\hbar}J_z = \frac{1}{2}\left[\frac{1}{\hbar}J_+ , \frac{1}{\hbar}J_-\right]$$
$$= \frac{1}{2}\left[y_+^\dagger y_- , y_+ y_-^\dagger\right]$$
$$= \frac{1}{2}\left(y_+^\dagger y_- y_+ y_-^\dagger - y_+ y_-^\dagger y_+^\dagger y_-\right)$$
$$= \frac{1}{2}\left(y_+^\dagger y_+ y_- y_-^\dagger - y_+ y_+^\dagger y_-^\dagger y_-\right)$$
$$= \frac{1}{2}\left\{y_+^\dagger y_+ (y_-^\dagger y_- + 1) - (y_+^\dagger y_+ + 1) y_-^\dagger y_-\right\}$$
$$= \frac{1}{2}\left(y_+^\dagger y_+ - y_-^\dagger y_-\right)$$

(5)

$$\frac{1}{\hbar^2}J^2 = \frac{1}{\hbar^2}J_+ J_- + \frac{1}{\hbar^2}J_z^2 - \frac{1}{\hbar}J_z$$
$$= y_+^\dagger y_- y_+ y_-^\dagger + \frac{1}{4}(y_+^\dagger y_+ - y_-^\dagger y_-)^2 - \frac{1}{2}(y_+^\dagger y_+ - y_-^\dagger y_-)$$
$$= y_+^\dagger y_+ (y_-^\dagger y_- + 1) + \frac{1}{4}(y_+^\dagger y_+ - y_-^\dagger y_-)^2 - \frac{1}{2}(y_+^\dagger y_+ - y_-^\dagger y_-)$$
$$= \frac{1}{4}(y_+^\dagger y_+ + y_-^\dagger y_-)^2 + \frac{1}{2}(y_+^\dagger y_+ + y_-^\dagger y_-)$$
$$= \left\{\frac{1}{2}(y_+^\dagger y_+ + y_-^\dagger y_-) + \frac{1}{2}\right\}^2 - \frac{1}{4}$$

Thus,

$$\sqrt{\frac{1}{\hbar^2}J^2 + \frac{1}{4}} - \frac{1}{2} = \frac{1}{2}(y_+^\dagger y_+ + y_-^\dagger y_-)$$

(6)

$$\frac{1}{\hbar}J_x = \frac{1}{2\hbar}(J_+ + J_-) = \frac{1}{2}(y_+^\dagger y_- + y_+ y_-^\dagger)$$

$$\frac{1}{\hbar}J_y = \frac{1}{2i\hbar}(J_+ - J_-) = \frac{1}{2i}(y_+^\dagger y_- - y_+ y_-^\dagger)$$

(7) Let us define

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \qquad \text{and} \qquad y^\dagger = \begin{pmatrix} y_+^\dagger & y_-^\dagger \end{pmatrix}$$

(8) Then, we observe,

$$\frac{1}{2} y^\dagger y = \frac{1}{2} \begin{pmatrix} y_+^\dagger & y_-^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = \frac{1}{2} \left(y_+^\dagger y_+ + y_-^\dagger y_- \right) = \sqrt{\frac{1}{\hbar^2} J^2 + \frac{1}{4}} - \frac{1}{2}$$

$$\frac{1}{2} y^\dagger \sigma_z y = \frac{1}{2} \begin{pmatrix} y_+^\dagger & y_-^\dagger \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = \frac{1}{2} \left(y_+^\dagger y_+ - y_-^\dagger y_- \right) = \frac{1}{\hbar} J_z$$

$$\frac{1}{2} y^\dagger \sigma_x y = \frac{1}{2} \begin{pmatrix} y_+^\dagger & y_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = \frac{1}{2} \left(y_+^\dagger y_- + y_+ y_-^\dagger \right) = \frac{1}{\hbar} J_x$$

$$\frac{1}{2} y^\dagger \sigma_y y = \frac{1}{2} \begin{pmatrix} y_+^\dagger & y_-^\dagger \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix} = \frac{1}{2i} \left(y_+^\dagger y_- - y_+ y_-^\dagger \right) = \frac{1}{\hbar} J_y$$

(9) Thus,

$$\frac{1}{\hbar^2} J^2 = \left(\frac{1}{2} y^\dagger y + \frac{1}{2} \right)^2 - \frac{1}{4}$$

$$\frac{1}{\hbar} \vec{J} = y^\dagger \, \frac{1}{2} \vec{\sigma} \, y$$

$$|j, m\rangle = |n_+, n_-\rangle = \frac{(y_+^\dagger)^{n_+}}{\sqrt{n_+!}} \frac{(y_-^\dagger)^{n_-}}{\sqrt{n_-!}} |0\rangle$$

$$= \frac{(y_+^\dagger)^{j+m}}{\sqrt{(j+m)!}} \frac{(y_-^\dagger)^{j-m}}{\sqrt{(j-m)!}} |0\rangle$$

(10) Notice that $\vec{J}\,|0\rangle = 0$ since $y_+ |0\rangle = 0$ and $y_- |0\rangle = 0$.