

Generating function for spherical harmonics

(1) Let us recollect.

(2) We presented the symbolic representation of harmonic oscillator:

$$[y_+ , y_+^\dagger] = 1$$

and constructed eigenvectors of the number operator, $N = y_+^\dagger y_+$,

$$N|n\rangle = n|n\rangle \qquad n = 0, 1, 2, \ldots$$

and found the properties

$$y|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$y^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle .$$

(3) Eigenvectors for angular momentum was constructed in terms of two harmonic oscillators,

$$\langle j, m| = \langle 0| \frac{y_+^{j+m}}{\sqrt{(j+m)!}} \frac{y_-^{j-m}}{\sqrt{(j-m)!}}$$

where

$$[y_+ , y_+^\dagger] = 1$$

$$[y_- , y_-^\dagger] = 1$$

$$[y_+ , y_-] = [y_+ , y_-^\dagger] = 0 .$$

(4) A rotation is described by an unitary transformation. In terms of Euler angles, (ψ, θ, ϕ), this rotation is described by

$$U(\psi, \theta, \phi) = e^{\frac{i}{\hbar}\psi J_z} \, e^{\frac{i}{\hbar}\theta J_y} \, e^{\frac{i}{\hbar}\phi J_z}$$

where J_x, J_y, J_z are generators of angular momentum.

(5) The eigenvectors of angular momentum transform under rotations as

$$\langle j, m | \, U(\psi, \theta, \phi) = \langle 0 | \frac{\bar{y}_+^{\,j+m}}{\sqrt{(j+m)!}} \frac{\bar{y}_-^{\,j-m}}{\sqrt{(j-m)!}}$$

where

$$\begin{pmatrix} \bar{y}_+ \\ \bar{y}_- \end{pmatrix} = \begin{bmatrix} e^{i\frac{\psi}{2}} \cos\frac{\theta}{2} \, e^{i\frac{\phi}{2}} & e^{i\frac{\psi}{2}} \sin\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \\ -e^{-i\frac{\psi}{2}} \sin\frac{\theta}{2} \, e^{i\frac{\phi}{2}} & e^{-i\frac{\psi}{2}} \cos\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \end{bmatrix} \begin{pmatrix} y_+ \\ y_- \end{pmatrix}$$

$$\begin{pmatrix} \bar{y}_+ \\ \bar{y}_- \end{pmatrix} \to \begin{pmatrix} \langle \frac{1}{2}, \frac{1}{2} | U \\ \langle \frac{1}{2}, -\frac{1}{2} | U \end{pmatrix} \qquad \begin{pmatrix} y_+ \\ y_- \end{pmatrix} \to \begin{pmatrix} \langle \frac{1}{2}, \frac{1}{2} | \\ \langle \frac{1}{2}, -\frac{1}{2} | \end{pmatrix}$$

which is a statement for the transformation properties of spin-$\frac{1}{2}$ particle.

(6) The rotation transformation function is obtained by projecting the eigenvector of angular momentum in the rotated frame with respect to the original (unrotated) frame.

That is,

$$\underset{\text{rotated eigenvector}}{\langle j,m|\,U(\psi,\theta,\phi)} = \sum_{j',m'} \underbrace{\langle j,m|\,U(\psi,\theta,\phi)\,|j',m'\rangle}_{\text{rotation transformation function}} \underset{\text{unrotated eigenvector}}{\langle j',m'|}$$

(7) We have seen that

$$\langle j,m|\,U(\psi,\theta,\phi)\,|j',m'\rangle = \delta_{jj'} \langle j,m|\,U(\psi,\theta,\phi)\,|j,m'\rangle$$

$$= \delta_{jj'}\, e^{im\psi} \langle j,m|\,U(\theta)\,|j,m'\rangle\, e^{im'\phi}$$

$$= \delta_{jj'}$$

(8) Using (5), (6), and (7) we found the generating function for $\langle j,m|U(\psi,\theta,\phi)|j',m'\rangle$ as.

$$e^{im\psi} \frac{\left(\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} y_+ + \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} y_-\right)^{j+m}}{\sqrt{(j+m)!}} \frac{\left(-\sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} y_+ + \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} y_-\right)^{j-m}}{\sqrt{(j-m)!}}$$

$$= \sum_{m'=-j}^{+j} \langle j,m|U(\psi,\theta,\phi)\,|j,m'\rangle \frac{y_+^{j+m'}}{\sqrt{(j+m')!}} \frac{y_-^{j-m'}}{\sqrt{(j-m')!}}$$

9 Let us now consider the special case of the projection of the

$m = 0$

eigenvector in the rotated basis, which necessarily requires j to be integer. Let $j = l = 0, 1, 2, \ldots$

10 For the special case of 9 we have, using 8,

$$\frac{1}{l!}\left(\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)^l \left(-\sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)^l$$

$$= \sum_{m'=-l}^{l} \langle l, 0 |\, U(\theta, \phi)\, | l, m' \rangle \frac{y_+^{\,l+m'}}{\sqrt{(l+m')!}} \frac{y_-^{\,l-m'}}{\sqrt{(l-m')!}}$$

10 $$\left(\cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)\left(-\sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}}\, y_+ + \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}}\, y_-\right)$$

$$= -\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\frac{\phi}{2}}\, y_+\right)^2 + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{-i\frac{\phi}{2}}\, y_-\right)^2 - \sin^2\frac{\theta}{2}\left(e^{i\frac{\phi}{2}}\, y_+\right)\left(e^{-i\frac{\phi}{2}}\, y_-\right)$$

$$+ \cos^2\frac{\theta}{2}\left(e^{i\frac{\phi}{2}}\, y_+\right)\left(e^{-i\frac{\phi}{2}}\, y_-\right)$$

$$= \frac{1}{2}\sin\theta\left(e^{-i\frac{\phi}{2}}\, y_-\right)^2 - \frac{1}{2}\sin\theta\left(e^{i\frac{\phi}{2}}\, y_+\right)^2 + \cos\theta\left(e^{i\frac{\phi}{2}}\, y_+\right)\left(e^{-i\frac{\phi}{2}}\, y_-\right)$$

$$= \sin\theta\cos\phi\,\frac{1}{2}\left(y_-^2 - y_+^2\right) + \sin\theta\sin\phi\,\frac{1}{2}\left(-i\,y_-^2 - i\,y_+^2\right) + \cos\theta\; y_+ y_-$$

(11) Introducing the vectors

$$\vec{r} = r\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec{a} = \frac{1}{2}\,(y_-^2 - y_+^2,\ -i\,y_-^2 - i\,y_+^2,\ 2\,y_+\,y_-)$$

(12) we can write

$$\frac{\vec{r}}{r}\cdot\vec{a} = \left(\cos\frac{\theta}{2}\,e^{i\frac{\phi}{2}}\,y_+ + \sin\frac{\theta}{2}\,e^{-i\frac{\phi}{2}}\,y_-\right)\left(-\sin\frac{\theta}{2}\,e^{i\frac{\phi}{2}}\,y_+ + \cos\frac{\theta}{2}\,e^{-i\frac{\phi}{2}}\,y_-\right)$$

$$= \frac{1}{2}\sin\theta\left(e^{-i\frac{\phi}{2}}\,y_-\right)^2 - \frac{1}{2}\sin\theta\left(e^{i\frac{\phi}{2}}\,y_+\right)^2 + \cos\theta\left(e^{i\frac{\phi}{2}}\,y_+\right)\left(e^{-i\frac{\phi}{2}}\,y_-\right)$$

(13) Using (12) the LHS of (10) can be written as.

$$\frac{1}{r^l}\frac{1}{l!}(\vec{r}\cdot\vec{a})^l = \frac{1}{2^l\,l!}\left[\sin\theta\left(e^{-i\frac{\phi}{2}}y_-\right)^2 - \sin\theta\left(e^{i\frac{\phi}{2}}y_+\right)^2 + 2\cos\theta\left(e^{i\frac{\phi}{2}}y_+\right)\left(e^{-i\frac{\phi}{2}}y_-\right)\right]^l$$

$$= \frac{1}{2^l\,l!}\left(e^{i\frac{\phi}{2}}y_+\right)^{2l}\left[\sin\theta\,\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)^2}{\left(e^{i\frac{\phi}{2}}y_+\right)^2} - \sin\theta + 2\cos\theta\,\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)}{\left(e^{i\frac{\phi}{2}}y_+\right)}\right]^l$$

$$= \frac{1}{2^l\,l!}\,\frac{\left(e^{i\frac{\phi}{2}}y_+\right)^{2l}}{(\sin\theta)^l}\left[\sin^2\theta\,\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)^2}{\left(e^{i\frac{\phi}{2}}y_+\right)^2} + 2\cos\theta\,\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)}{\left(e^{i\frac{\phi}{2}}y_+\right)} - \sin^2\theta\right]^l$$

$$= \frac{1}{2^l\,l!}\,\frac{\left(e^{i\frac{\phi}{2}}y_+\right)^{2l}}{(\sin\theta)^l}\left[\left\{\sin\theta\,\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)}{\left(e^{i\frac{\phi}{2}}y_+\right)} + \cos\theta\right\}^2 - 1\right]^l$$

(14) Consider

$$I_l(t_0+t) = \left[(t_0+t)^2-1\right]^l$$

Using

$$I_l(t_0+t) = e^{t_0 \frac{d}{dt}} I_l(t)$$

we can write

$$I_l(t_0+t) = e^{t_0\frac{d}{dt}} (t^2-1)^l$$

$$= \sum_{n=0}^{\infty} \frac{t_0^n}{n!}\left(\frac{d}{dt}\right)^n (t^2-1)^l$$

$$= \sum_{n=0}^{2l} \frac{t_0^n}{n!}\left(\frac{d}{dt}\right)^n (t^2-1)^l \qquad \begin{matrix} n = l-m \\ m = l-n \end{matrix}$$

$$= \sum_{m=+l}^{-l} \frac{t_0^{l-m}}{(l-m)!}\left(\frac{d}{dt}\right)^{l-m} (t^2-1)^l$$

$$= \sum_{m=-l}^{+l} \frac{t_0^{l-m}}{(l-m)!}\left(\frac{d}{dt}\right)^{l-m} (t^2-1)^l \qquad \text{(by rearranging the terms.)}$$

(15) Using (14) in (13)

$$\frac{1}{r^l}\frac{1}{l!}(\vec{r}\cdot\vec{a})^l = \frac{1}{2^l l!}\frac{\left(e^{i\frac{\phi}{2}}y_+\right)^{2l}}{(\sin\theta)^l}\sum_{m=-l}^{l}\frac{1}{(l-m)!}\left\{\sin\theta\frac{\left(e^{-i\frac{\phi}{2}}y_-\right)}{\left(e^{i\frac{\phi}{2}}y_+\right)}\right\}^{l-m}\left(\frac{d}{dt}\right)^{l-m}(t^2-1)^l\Bigg|_{t=\cos\theta}$$

$$= \sum_{m=-l}^{l}\frac{1}{(\sin\theta)^m}\frac{1}{(l-m)!}\left(e^{i\frac{\phi}{2}}y_+\right)^{l+m}\left(e^{-i\frac{\phi}{2}}y_-\right)^{l-m}\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l l!}\Bigg|_{t=\cos\theta}$$

$$= \sum_{m=-l}^{l} e^{im\phi}\frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l l!}\Bigg|_{t=\cos\theta}$$

16 Comparing 10, 12, and 15 we can read out

$$\langle l,0|\,U(\theta,\phi)\,|l,m\rangle = e^{im\phi}\,\frac{1}{(\sin\theta)^m}\sqrt{\frac{(l+m)!}{(l-m)!}}\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l\,l!}\Bigg|_{t=\cos\theta}$$

17 Define the spherical harmonics as

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,\langle l,0|\,U(\theta,\phi)\,|l,m\rangle$$

$$= \sqrt{\frac{2l+1}{4\pi}}\,e^{im\phi}\,\frac{1}{(\sin\theta)^m}\sqrt{\frac{(l+m)!}{(l-m)!}}\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l\,l!}\Bigg|_{t=\cos\theta}$$

18 For $m=0$ we have

$$Y_{l0}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\theta)$$

where $P_l(t)$ are the Legendre polynomials

$$P_l(t) = \left(\frac{d}{dt}\right)^l\frac{(t^2-1)^l}{2^l\,l!}.$$

(19) Thus, we have the generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, as

$$\frac{1}{l!}\left(\frac{\vec{r}}{r}\cdot\vec{a}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\; Y_{lm}(\theta,\phi)\;\psi_{lm}$$

where

$$\vec{r} = r\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec{a} = \frac{1}{2}\,(y_-^2 - y_+^2,\ -i\,y_-^2 - i\,y_+^2,\ 2\,y_+ y_-)$$

$$\psi_{lm} = \frac{y_+^{\,l+m}}{\sqrt{(l+m)!}}\;\frac{y_-^{\,l-m}}{\sqrt{(l-m)!}}$$

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\;\frac{1}{(\sin\theta)^m}\sqrt{\frac{(l+m)!}{(l-m)!}}\;\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l\,l!}\Bigg|_{t=\cos\theta},$$

which is a restatement of

$$\frac{1}{l!}\,(\bar{y}_+\,\bar{y}_-)^l = \sum_{m'=-l}^{l}\langle l,0|\,U(\theta,\phi)\,|l,m'\rangle\;\frac{y_+^{\,l+m'}}{\sqrt{(l+m')!}}\;\frac{y_-^{\,l-m'}}{\sqrt{(l-m')!}}$$

where

$$\begin{pmatrix}\bar{y}_+\\ \bar{y}_-\end{pmatrix} = \begin{bmatrix}\cos\frac{\theta}{2}\,e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}\,e^{-i\frac{\phi}{2}}\\ -\sin\frac{\theta}{2}\,e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}\,e^{-i\frac{\phi}{2}}\end{bmatrix}\begin{pmatrix}y_+\\ y_-\end{pmatrix}.$$