SET-12

Orthonormality of spherical harmonics

(1) The generating function of spherical harmonics is

$$\frac{1}{l!}\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\; Y_{lm}(\theta,\phi)\;\psi_{lm}$$

where

$$\vec{r} = r\,(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\vec{a} = \frac{1}{2}\,(y_-^2 - y_+^2,\ -iy_-^2 - iy_+^2,\ 2y_+y_-)$$

and

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\; e^{im\phi}\sqrt{\frac{(l+m)!}{(l-m)!}}\;\frac{1}{(\sin\theta)^m}\left(\frac{d}{dt}\right)^{l-m}\frac{(t^2-1)^l}{2^l\, l!}\Bigg|_{t=\cos\theta}$$

$$\psi_{lm} = \frac{y_+^{\,l+m}}{\sqrt{(l+m)!}}\;\frac{y_-^{\,l-m}}{\sqrt{(l-m)!}}$$

(2) Our goal will be to derive the orthonormality condition,

$$\int d\Omega\; Y_{lm}(\theta,\phi)\; Y_{l'm'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}.$$

where

$$d\Omega = d\phi\,\sin\theta\, d\theta \qquad 0\le\phi<2\pi,\quad 0\le\theta<\pi.$$

(3) Using (1), and after taking the complex conjugate we obtain

$$\frac{1}{l!}\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{l} = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\; Y_{lm}(\theta,\phi)\;\psi_{lm}$$

and

$$\frac{1}{l!}\left(\vec{a}^{*}\cdot\frac{\vec{r}}{r}\right)^{l} = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\; Y_{lm}^{*}(\theta,\phi)\;\psi_{lm}^{*}$$

(4) Multiplying the two expressions in (3) and integrating

$$\frac{1}{l!}\,\frac{1}{l'!}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\;\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{l}\left(\vec{a}^{*}\cdot\frac{\vec{r}}{r}\right)^{l'}$$

$$= \sum_{m=-l}^{l}\sum_{m'=-l'}^{l'}\sqrt{\frac{4\pi}{2l+1}}\sqrt{\frac{4\pi}{2l'+1}}\;\psi_{l'm'}^{*}\left[\int d\Omega\; Y_{l'm'}^{*}(\theta,\phi)\,Y_{lm}(\theta,\phi)\right]\psi_{lm}$$

(5) Let us define the left hand side as

$$I_{ll'}(\vec{a},\vec{a}^{*}) = \frac{1}{l!}\,\frac{1}{l'!}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\, d\theta\;\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{l}\left(\vec{a}^{*}\cdot\frac{\vec{r}}{r}\right)^{l'}$$

(6) Since $I_{ll'}(\vec{a},\vec{a}^{*})$ is built out of $\vec{a}$ and $\vec{a}^{*}$, and because $\vec{a}\cdot\vec{a}=0$, $\vec{a}^{*}\cdot\vec{a}^{*}=0$, and $\vec{a}\cdot\vec{a}^{*}\neq 0$, it necessarily requires $l=l'$. Thus we necessarily require the form

$$I_{ll'} = \delta_{ll'}\,(\vec{a}\cdot\vec{a}^{*})^{l}\; C$$

7 Using 6 in 5 we have.

$$\frac{1}{l!}\frac{1}{l'!}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^l \left(\vec{a}^*\cdot\frac{\vec{r}}{r}\right)^{l'} = \delta_{ll'} \left(\vec{a}\cdot\vec{a}^*\right)^l \, c$$

8 To determine c we consider a special value for $\vec{a}$.

$$\vec{a} = (1, i, 0) \qquad \vec{a}\cdot\vec{a} = 0$$

$$\vec{a}^* = (1, -i, 0) \qquad \vec{a}^*\cdot\vec{a}^* = 0$$

$$\vec{a}\cdot\vec{a}^* = 2$$

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$$\vec{a}\cdot\frac{\vec{r}}{r} = \sin\theta\cos\phi + i\sin\theta\sin\phi = \sin\theta\, e^{i\phi}$$

$$\vec{a}^*\cdot\frac{\vec{r}}{r} = \sin\theta\cos\phi - i\sin\theta\sin\phi = \sin\theta\, e^{-i\phi}$$

10 Using 8 and 9 in 7

$$\frac{1}{(l!)^2}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \left(\sin\theta\, e^{i\phi}\right)^l \left(\sin\theta\, e^{-i\phi}\right)^l = 2^l c$$

$$c = \frac{2\pi}{2^l (l!)^2}\int_0^{\pi} \sin\theta\, d\theta \left(\sin^2\theta\right)^l$$

$$= \frac{2\pi}{2^l (l!)^2}\int_{-1}^{+1} dt\, (1-t^2)^l$$

$$= \frac{2\pi}{2^l (l!)^2}\int_1^0 -2\,dx\; 2^{2l}\, x^l (1-x)^l$$

$$= \frac{4\pi\, 2^l}{(l!)^2}\int_0^1 dx\; x^l (1-x)^l$$

$$= \frac{4\pi\, 2^l}{(l!)^2}\,\frac{(l!)^2}{(2l+1)!} = \frac{4\pi}{2l+1}\,\frac{2^l}{(2l)!}$$

$$(1-t^2)^l = (1-t)^l (1+t)^l$$

$$x = \frac{1-t}{2} \qquad dx = -\frac{dt}{2}$$

$$1-2x = t$$

$$\frac{1+t}{2} = 1-x$$

$$B(m,n) = \int_0^1 dx\; x^{m-1}(1-x)^{n-1} = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

11 Using 10 in 7

$$\frac{1}{l!}\,\frac{1}{l'!}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta \,\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{l}\left(\vec{a}^{*}\cdot\frac{\vec{r}}{r}\right)^{l'} = \delta_{ll'}\,(\vec{a}\cdot\vec{a}^{*})^{l}\,\frac{4\pi}{2l+1}\,\frac{2^{l}}{(2l)!}$$

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$$\vec{a} = \frac{1}{2}\left(y_-^2 - y_+^2,\ -i y_-^2 - i y_+^2,\ 2 y_+ y_-\right)$$

$$\vec{a}^{*} = \frac{1}{2}\left(y_-^{*2} - y_+^{*2},\ i y_-^{*2} + i y_+^{*2},\ 2 y_+^{*} y_-^{*}\right)$$

$$\vec{a}\cdot\vec{a}^{*} = \frac{1}{4}\left(y_-^2 - y_+^2\right)\left(y_-^{*2} - y_+^{*2}\right) + \frac{1}{4}\left(y_-^2 + y_+^2\right)\left(y_-^{*2} + y_+^{*2}\right) + y_+ y_+^{*} y_- y_-^{*}$$

$$= \frac{1}{2}\left(y_- y_-^{*}\right)^2 + \frac{1}{2}\left(y_+ y_+^{*}\right)^2 + \left(y_- y_-^{*}\right)\left(y_+ y_+^{*}\right)$$

$$= \frac{1}{2}\left(y_- y_-^{*} + y_+ y_+^{*}\right)^2$$

$$(x+y)^N = \sum_{n=0}^{N} x^n y^{N-n}\,\frac{N!}{n!\,(N-n)!}$$

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$$(\vec{a}\cdot\vec{a}^{*})^{l} = \frac{1}{2^{l}}\left(y_- y_-^{*} + y_+ y_+^{*}\right)^{2l}$$

$$= \frac{1}{2^{l}}\sum_{n=0}^{2l}\left(y_- y_-^{*}\right)^{n}\left(y_+ y_+^{*}\right)^{2l-n}\frac{(2l)!}{n!\,(2l-n)!}$$

$n = l - m$

$$= \frac{1}{2^{l}}\sum_{m=+l}^{-l}\left(y_- y_-^{*}\right)^{l-m}\left(y_+ y_+^{*}\right)^{l+m}\frac{(2l)!}{(l-m)!\,(l+m)!}$$

$$= \frac{(2l)!}{2^{l}}\sum_{m=-l}^{l}\frac{(y_+^{*})^{l+m}}{\sqrt{(l+m)!}}\,\frac{(y_-^{*})^{l-m}}{\sqrt{(l-m)!}}\,\frac{y_+^{l+m}}{\sqrt{(l+m)!}}\,\frac{y_-^{l-m}}{\sqrt{(l-m)!}}$$

$$= \frac{(2l)!}{2^{l}}\sum_{m=-l}^{l}\psi_{lm}^{*}\,\psi_{lm}.$$

14 Using 13 in 11 we have

$$\frac{1}{l!}\,\frac{1}{l'!}\int d\Omega\,\left(\vec{a}^{*}\cdot\frac{\vec{r}}{r}\right)^{l'}\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^{l} = \delta_{ll'}\,\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\psi_{lm}^{*}\,\psi_{lm}$$

$$= \sqrt{\frac{4\pi}{2l+1}}\,\sqrt{\frac{4\pi}{2l'+1}}\,\sum_{m=-l}^{l}\,\sum_{m'=-l'}^{l'}\psi_{l'm'}^{*}\left[\,\delta_{ll'}\,\delta_{mm'}\,\right]\psi_{lm}$$

15 Comparing 4 and 14 we have the orthonormality condition

$$\int d\Omega\; Y_{l'm'}^{*}(\theta,\phi)\;Y_{lm}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'}.$$

16 We also state the completeness relation

$$\sum_{l=0}^{\infty}\,\sum_{m=-l}^{l} Y_{lm}^{*}(\theta,\phi)\;Y_{lm}(\theta',\phi') = \delta(\theta-\theta')\,\frac{\delta(\phi-\phi')}{\sin\theta}.$$