Differential equation of spherical harmonics : Realization of angular momentum states.

① In terms of operators we have the generating function

$$\langle 0| \frac{1}{l!}\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)^l = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}}\; Y_{lm}(\theta,\phi)\; \langle 0|\, \Psi_{lm}$$

where

$$\vec{a}\cdot\frac{\vec{r}}{r} = \bar{y}_+\, \bar{y}_-$$

$$\vec{a} = \frac{1}{2}\left(y_-^2 - y_+^2 ,\; -i y_-^2 - i y_+^2 ,\; 2 y_+ y_- \right)$$

$$\vec{r} = r\left(\sin\theta\cos\phi ,\; \sin\theta\sin\phi ,\; \cos\theta\right)$$

and

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}}\; e^{im\phi} \sqrt{\frac{(l+m)!}{(l-m)!}}\; \frac{1}{(\sin\theta)^m}\left(\frac{d}{dt}\right)^{l-m} \frac{(t^2-1)^l}{2^l\, l!}\Bigg|_{t=\cos\theta}$$

$$\Psi_{lm} = \frac{y_+^{\,l+m}}{\sqrt{(l+m)!}}\; \frac{y_-^{\,l-m}}{\sqrt{(l-m)!}}$$

② Is $\vec{a}$ really a vector ?

③ We note that $\vec{a}$ can be expressed as

$$\vec{a} = \frac{1}{i}\; y^T\, \sigma_y\, \frac{1}{2}\vec{\sigma}\; y$$

where $y^T = (y_+ , y_-)$ and $\vec{\sigma}$ are Pauli matrices.

3 We also note that $\sigma_i \sigma_j + \sigma_j \sigma_i = \delta_{ij}$

$$\sigma_x^T = \sigma_x \qquad \sigma_x^T \sigma_y = \sigma_x \sigma_y = -\sigma_y \sigma_x$$

$$\sigma_y^T = -\sigma_y \qquad \sigma_y^T \sigma_y = -\sigma_y \sigma_y = -\sigma_y \sigma_y$$

$$\sigma_z^T = \sigma_z \qquad \sigma_z^T \sigma_z = \sigma_z \sigma_y = -\sigma_y \sigma_z$$

$$\Rightarrow \quad \vec{\sigma}^T \sigma_y = -\sigma_y \vec{\sigma}$$

$$\vec{\sigma}^T = -\sigma_y \vec{\sigma} \sigma_y$$

4 To verify the vector nature of $\vec{a}$ we investigate the transformat of $\vec{a}$ under infinitesmal rotations.

$$\vec{a}' = \frac{1}{i} (y u)^T \sigma_y \frac{1}{2} \vec{\sigma} \, u \, y$$

where

$$u = 1 + i \, \delta\vec{\omega} \cdot \frac{1}{2} \vec{\sigma} .$$

Note that the Pauli matrices in the above structure appear in the form of matrix representation, $\langle \sigma | \vec{\sigma} | \sigma' \rangle$, which does not transform under rotations.

5
$$\vec{a}' = \frac{1}{i}\, y^T u^T \sigma_y \frac{1}{2}\vec{\sigma}\, u\, y$$
$$= \frac{1}{i}\, y^T \left\{1 + i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right\}^T \sigma_y \frac{1}{2}\vec{\sigma} \left\{1 + i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right\} y$$
$$= \frac{1}{i}\, y^T \left\{1 + i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}^T\right\} \sigma_y \frac{1}{2}\vec{\sigma} \left\{1 + i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right\} y$$
$$= \vec{a} + \frac{1}{i}\, y^T \left[\sigma_y \frac{1}{2}\vec{\sigma}\left(i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right) + \left(i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}^T\right)\sigma_y \frac{1}{2}\vec{\sigma}\right] y$$
$$= \vec{a} + \frac{1}{i}\, y^T \left[\sigma_y \frac{1}{2}\vec{\sigma}\left(i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right) - \sigma_y\left(i\delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right)\frac{1}{2}\vec{\sigma}\right] y$$
$$= \vec{a} + \frac{1}{i}\, y^T \sigma_y\, i\left[\frac{1}{2}\vec{\sigma},\ \delta\vec{\omega}\cdot\frac{1}{2}\vec{\sigma}\right] y$$
$$= \vec{a} - \frac{1}{i}\, y^T \sigma_y \left(\delta\vec{\omega}\times\frac{1}{2}\vec{\sigma}\right) y$$
$$= \vec{a} - \delta\vec{\omega}\times\vec{a}$$

Thus, $\vec{a}$ transforms like a vector.

6 Thus,
$$\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)'$$
transforms like a scalar, ie., does not change.
$$\delta\left[\left(\vec{a}\cdot\frac{\vec{r}}{r}\right)'\right] = 0$$

$\langle 0 | \Psi_{lm} = \langle l, m|$

(7) Using (6) in (1) we conclude.

$$\delta \left[\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \langle l,m| \right] = 0$$

$$\sum_{m=-l}^{l} \left[\delta Y_{lm}(\theta,\phi) \right] \langle l,m| + \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \left[\delta \langle l,m| \right] = 0$$

$\vec{r}\,' = \vec{r} - \delta\vec{\omega} \times \vec{r}$.

(8)
$$\begin{aligned} \delta Y_{lm}(\theta,\phi) &= \delta\vec{r} \cdot \vec{\nabla} Y_{lm}(\theta,\phi) \\ &= -(\delta\vec{\omega} \times \vec{r}) \cdot \vec{\nabla} Y_{lm}(\theta,\phi) \\ &= -\delta\vec{\omega} \cdot \vec{r} \times \vec{\nabla} Y_{lm}(\theta,\phi) \end{aligned}$$

$$\delta \langle l,m| = \langle l,m| \frac{i}{\hbar} \delta\vec{\omega} \cdot \vec{L}$$

(9) Using (8) in (7)

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \langle l,m| \frac{i}{\hbar} \delta\vec{\omega} \cdot \vec{L} = \sum_{m=-l}^{l} \delta\vec{\omega} \cdot \vec{r} \times \vec{\nabla} Y_{lm}(\theta,\phi) \langle l,m|$$

(10) Since it is true for arbitrary $\delta\vec{\omega}$ we have.

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \langle l,m| \frac{i}{\hbar} \vec{L} = \sum_{m=-l}^{l} (\vec{r} \times \vec{\nabla}) Y_{lm}(\theta,\phi) \langle l,m|$$

(11)

$$\vec{r} \times \vec{\nabla} = r\,\hat{r} \times \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\,\partial\theta} + \hat{\phi}\frac{1}{r\,\text{Sin}\,\theta}\frac{\partial}{\partial\phi} \right]$$

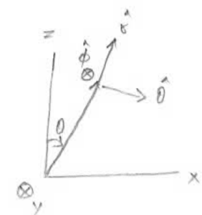
$$= (\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta} + (\hat{r}\times\hat{\phi})\frac{1}{\text{Sin}\,\theta}\frac{\partial}{\partial\phi}$$

(12)

$$\hat{r} = \frac{(\vec{\nabla} r)}{|\vec{\nabla} r|} = (\text{Sin}\,\theta\,\text{Cos}\,\phi,\ \text{Sin}\,\theta\,\text{Sin}\,\phi,\ \text{Cos}\,\theta)$$

$$\hat{\theta} = \frac{(\vec{\nabla}\theta)}{|\vec{\nabla}\theta|} = (\text{Cos}\,\theta\,\text{Cos}\,\phi,\ \text{Cos}\,\theta\,\text{Sin}\,\phi,\ -\text{Sin}\,\theta)$$

$$\hat{\phi} = \frac{(\vec{\nabla}\phi)}{|\vec{\nabla}\phi|} = (-\text{Sin}\,\phi,\ \text{Cos}\,\phi,\ 0)$$



(13)

$$\vec{r} \times \vec{\nabla} = \hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\text{Sin}\,\theta}\frac{\partial}{\partial\phi}$$

$$= \left[-\text{Sin}\,\phi\,\hat{x} + \text{Cos}\,\phi\,\hat{y}\right]\frac{\partial}{\partial\theta}$$

$$- \left[\text{Cos}\,\theta\,\text{Cos}\,\phi\,\hat{x} + \text{Cos}\,\theta\,\text{Sin}\,\phi\,\hat{y} - \text{Sin}\,\theta\,\hat{z}\right]\frac{1}{\text{Sin}\,\theta}\frac{\partial}{\partial\phi}$$

$$= \hat{x}\left[-\text{Sin}\,\phi\frac{\partial}{\partial\theta} - \text{Cos}\,\phi\,\text{Cot}\,\theta\frac{\partial}{\partial\phi}\right]$$

$$+ \hat{y}\left[\text{Cos}\,\phi\frac{\partial}{\partial\theta} - \text{Sin}\,\phi\,\text{Cot}\,\theta\frac{\partial}{\partial\phi}\right]$$

$$+ \hat{z}\frac{\partial}{\partial\phi}$$

(14) $$(\vec{r} \times \vec{\nabla}) \cdot (\vec{r} \times \vec{\nabla}) = \left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right)\left(-\sin\phi \frac{\partial}{\partial\theta} - \cos\phi \cot\theta \frac{\partial}{\partial\phi}\right)$$

$$+ \left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right)\left(+\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi}\right) + \frac{\partial^2}{\partial\phi^2}$$

$$= \frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot^2\theta \frac{\partial^2}{\partial\phi^2} - \cos\phi \sin\phi \cot^2\theta \frac{\partial}{\partial\phi} + \sin\phi \cos\phi \cot^2\theta \frac{\partial}{\partial\phi}$$

$$+ \sin\phi \cos\phi \frac{\partial}{\partial\phi} \frac{\partial}{\partial\theta} \cot\theta - \sin\phi \cos\phi \frac{\partial}{\partial\phi} \frac{\partial}{\partial\theta} \cot\theta$$

$$+ \sin\phi \cos\phi \cot\theta \frac{\partial}{\partial\phi} \frac{\partial}{\partial\theta} - \sin\phi \cos\phi \cot\theta \frac{\partial}{\partial\phi} \frac{\partial}{\partial\theta}$$

$$+ \cot\theta \frac{\partial}{\partial\theta}$$

$$= \frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot^2\theta \frac{\partial^2}{\partial\phi^2} + \cot\theta \frac{\partial}{\partial\theta}$$

$$= \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta}$$

(16) Using (10) we have, taking $\hat{z}$ component,

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \langle l,m| L_z = \sum_{m=-l}^{l} \frac{\hbar}{i} \frac{\partial}{\partial\phi} Y_{lm}(\theta,\phi) \langle l,m|$$

$$\sum_{m=-l}^{l} \hbar\, m\, Y_{lm}(\theta,\phi) \langle l,m| = \sum_{m=-l}^{l} \frac{\hbar}{i} \frac{\partial}{\partial\phi} Y_{lm}(\theta,\phi) \langle l,m|$$

Thus, using completeness of $\langle l,m|$ implies.

$$\frac{1}{i} \frac{\partial}{\partial\phi} Y_{lm}(\theta,\phi) = m\, Y_{lm}(\theta,\phi)$$

17 Using 10 we can write

$$\left[\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \ \langle l,m| \right] \vec{L} = \frac{\hbar}{i}(\vec{r}\times\vec{\nabla}) \left[\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \ \langle l,m| \right]$$

18

$$L^2 \ |l,m\rangle = \hbar^2 \, l(l+1) \ |l,m\rangle$$

$$L_z \ |l,m\rangle = \hbar m \ |l,m\rangle$$

$$L_+ \ |l,m\rangle = \sqrt{(l-m)(l+m+1)} \ |l,m+1\rangle$$

$$L_- \ |l,m\rangle = \sqrt{(l+m)(l-m+1)} \ |l,m-1\rangle$$

$L_+ = L_x + iL_y$

$L_- = L_x - iL_y$

19

$$\langle l,m| \ L^2 = \hbar^2 \, l(l+1) \ \langle l,m|$$

$$\langle l,m| \ L_z = \hbar m \ \langle l,m|$$

$$\langle l,m| \ L_- = \sqrt{(l-m)(l+m+1)} \ \langle l,m+1|$$

$$\langle l,m| \ L_+ = \sqrt{(l+m)(l-m+1)} \ \langle l,m-1|$$

20 Using 19 in 17

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \ \langle l,m| \ L_+ = \frac{\hbar}{i}\left\{ \hat{x}\cdot\vec{r}\times\vec{\nabla} + i\,\hat{y}\cdot\vec{r}\times\vec{\nabla} \right\} \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \ \langle l,m|$$

$$\sum_{m=-l+1}^{l} Y_{lm}(\theta,\phi)\,\hbar\sqrt{(l+m)(l-m+1)} \ \langle l,m-1| = \frac{\hbar}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} + i\,\hat{y}\cdot\vec{r}\times\vec{\nabla} \right] \sum_{m=-l}^{l} Y_{lm}(\theta,\phi) \ \langle l,m|$$

because $\langle l,-l| \ L_+ = 0$

(21) $$\sum_{m=-l+1}^{l} Y_{lm}(\theta,\phi)\sqrt{(l+m)(l-m+1)}\ \langle l, m-1|$$

$$m-1 = m'$$

$$= \sum_{m'=-l}^{l-1} Y_{l,m'+1}(\theta,\phi)\sqrt{(l-m')(l+m'+1)}\ \langle l, m'|$$

$$= \sum_{m'=-l}^{l} Y_{l,m'+1}(\theta,\phi)\sqrt{(l-m')(l+m'+1)}\ \langle l, m'|$$ — involves adding 0. because at $m'=l$, square root is 0.

(22) Using (21) in (20)

$$\sum_{m=-l}^{l} Y_{l,m+1}(\theta,\phi)\sqrt{(l-m)(l+m+1)}\ \langle l,m| = \frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} + i\,\hat{y}\cdot\vec{r}\times\vec{\nabla}\right]\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\ \langle l,m|$$

Thus, using completeness of $\langle l,m|$ we have.

$$\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} + i\,\hat{y}\cdot(\vec{r}\times\vec{\nabla})\right]Y_{lm}(\theta,\phi) = \sqrt{(l-m)(l+m+1)}\ Y_{l,m+1}(\theta,\phi)$$

(23) Similarly we will have

$$\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} - i\,\hat{y}\cdot(\vec{r}\times\vec{\nabla})\right]Y_{lm}(\theta,\phi) = \sqrt{(l+m)(l-m+1)}\ Y_{l,m-1}(\theta,\phi)$$

24 Using 13 we have.

$$\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} \pm i\,\hat{y}\cdot\vec{r}\times\vec{\nabla}\right] = -i\,\hat{x}\cdot\vec{r}\times\vec{\nabla} \pm \hat{y}\cdot\vec{r}\times\vec{\nabla}$$

$$= -i\left[-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right] \pm \left[\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= (\pm\cos\phi + i\sin\phi)\frac{\partial}{\partial\theta} + (i\cos\phi \mp \sin\phi)\cot\theta\frac{\partial}{\partial\phi}$$

$$= \pm(\cos\phi \pm i\sin\phi)\frac{\partial}{\partial\theta} + i(\cos\phi \pm i\sin\phi)\cot\theta\frac{\partial}{\partial\phi}$$

$$= e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= \frac{1}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right]$$

25 Using 16, 22 and 23

$$\frac{1}{i}\hat{z}\cdot\vec{r}\times\vec{\nabla}\,Y_{lm}(\theta,\phi) = \frac{1}{i}\frac{\partial}{\partial\phi}Y_{lm}(\theta,\phi) = m\,Y_{lm}(\theta,\phi)$$

and

$$\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} \pm i\,\hat{y}\cdot\vec{r}\times\vec{\nabla}\right]Y_{lm}(\theta,\phi) = \frac{1}{i}e^{\pm i\phi}\left[\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right]Y_{l,m}(\theta,\phi)$$

$$= \sqrt{(l\mp m)(l\pm m+1)}\;Y_{l,m\pm1}(\theta,\phi)$$

26 Using 25 we have.

$$\frac{1}{i}\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} + i\hat{y}\cdot\vec{r}\times\vec{\nabla}\right]\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} - i\hat{y}\cdot\vec{r}\times\vec{\nabla}\right] Y_{lm}(\theta,\phi)$$

$$= \frac{1}{i} e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\frac{1}{i}e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) Y_{lm}(\theta,\phi)$$

$$= -\left[\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) + i\cot\theta\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{\partial^2}{\partial\theta^2} - i\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\phi} + i\cot\theta\frac{\partial}{\partial\phi}\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta} - i\cot^2\theta\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{\partial^2}{\partial\theta^2} + i\,\mathrm{cosec}^2\theta\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta} - i\cot^2\theta\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2} - \frac{1}{i}\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$= - \Big[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2}$$

which should be compared to

$$L_+ L_- = L_x^2 + L_y^2 + L_z.$$

Further, on the right we have

$$\sqrt{(l+m)(l-m+1)}\ \sqrt{[l-(m-1)]\,[l+(m-1)+1]} = \sqrt{(l+m)(l-m+1)(l-m+1)(l+m)}$$

$$= (l+m)(l-m+1)$$

$$= l(l+1) - m^2 - ml + m(l+1)$$

$$= l(l+1) - m^2 + m$$

27 Using 25 we also have.

$$\frac{1}{i}\frac{1}{i}\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} - i\,\hat{y}\cdot\vec{r}\times\vec{\nabla}\right]\left[\hat{x}\cdot\vec{r}\times\vec{\nabla} + i\,\hat{y}\cdot\vec{r}\times\vec{\nabla}\right] Y_{lm}(\theta,\phi)$$

$$= \frac{1}{i}\, e^{-i\phi}\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\frac{1}{i}\, e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) Y_{lm}(\theta,\phi)$$

$$= -\left[\left(-i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right) - i\cot\theta\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{\partial^2}{\partial\theta^2} + i\frac{\partial}{\partial\theta}\cot\theta\frac{\partial}{\partial\phi} - i\cot\theta\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta} + i\cot^2\theta\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} - i\,\mathrm{cosec}^2\theta\frac{\partial}{\partial\phi} + i\cot^2\theta\frac{\partial}{\partial\phi} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - i\frac{\partial}{\partial\phi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi)$$

$$= -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial\phi^2} + \frac{1}{i}\frac{\partial}{\partial\phi}\right] Y_{lm}(\theta,\phi)$$

which should be compared with

$$L_-L_+ = L_x^2 + L_y^2 - L_z.$$

and on the right we have.

$$\sqrt{(l-m)(l+m+1)}\,\sqrt{[l+(m+1)][l-(m+1)+1]} = \sqrt{(l-m)(l+m+1)(l+m+1)(l-m)}$$

$$= (l-m)(l+m+1)$$

$$= l(l+1) - m^2 + ml - m(l+1)$$

$$= l(l+1) - m^2 - m$$

28 Summary :

L_z : $\quad \frac{1}{i}\,\hat{z}\cdot(\vec{r}\times\vec{\nabla})\;Y_{lm}(\theta,\phi) = \frac{1}{i}\frac{\partial}{\partial\phi}\;Y_{lm}(\theta,\phi)$

$$= m\;Y_{lm}(\theta,\phi)$$

$L_\pm$: $\quad \frac{1}{i}\left[\,\hat{x}\cdot(\vec{r}\times\vec{\nabla}) \pm i\,\hat{y}\cdot(\vec{r}\times\vec{\nabla})\right]\;Y_{lm}(\theta,\phi) = \frac{1}{i}\;e^{\pm i\phi}\left(\pm i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\;Y_{lm}(\theta,\phi)$

$$= \sqrt{(l\mp m)(l\pm m+1)}\quad Y_{l,\,m\pm1}(\theta,\phi)$$

$L_x^2+L_y^2$: $\quad \frac{1}{2}\left(L_+L_- + L_-L_+\right)\;Y_{lm}(\theta,\phi) = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right)\;Y_{lm}(\theta,\phi)$

$$= \left[\,l(l+1) - m^2\right]\;Y_{lm}(\theta,\phi)$$

L_z^2 : $\quad \frac{1}{i}\frac{1}{i}\left(\hat{z}\cdot\vec{r}\times\vec{\nabla}\right)^2\;Y_{lm}(\theta,\phi) = -\frac{\partial^2}{\partial\phi^2}\;Y_{lm}(\theta,\phi)$

$$= m^2\;Y_{lm}(\theta,\phi)$$

L^2 : $\quad \frac{1}{i}\frac{1}{i}\left(\vec{r}\times\vec{\nabla}\right)\cdot\left(\vec{r}\times\vec{\nabla}\right)\;Y_{lm}(\theta,\phi) = -\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\;Y_{lm}(\theta,\phi)$

$$= l(l+1)\;Y_{lm}(\theta,\phi)\,.$$