Midterm Exam No. 02 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Date: 2014 Apr 2

1. (**20 points.**) From Maxwell's equations, without introducing potentials, derive

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{B}(\mathbf{r},t) = \mu_0 \boldsymbol{\nabla}\times\mathbf{J}(\mathbf{r},t). \tag{1a}$$

2. (**20 points.**) Using the identity

$$\delta(F(x)) = \sum_r \frac{\delta(x-a_r)}{\left|\frac{dF}{dx}\Big|_{x=a_r}\right|}, \tag{2}$$

where the sum on r runs over the roots a_r of the equation $F(x) = 0$, evaluate

$$\delta(ax^2 + bx + c). \tag{3}$$

3. (**20 points.**) Consider the retarded Green's function

$$G(\mathbf{r}-\mathbf{r}', t-t') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t - t' - \frac{1}{c}|\mathbf{r}-\mathbf{r}'|\right). \tag{4}$$

   (a) For $\mathbf{r}' = 0$ and $t' = 0$ show that

$$G(r,t) = \frac{c}{r}\delta(r - ct). \tag{5}$$

   (b) Then, evalauate

$$\int_{-\infty}^{\infty} dt\, G(r,t). \tag{6}$$

   (c) From the answer above, what can you comment on the physical interpretation of $\int_{-\infty}^{\infty} dt\, G(r,t)$.

4. (**20 points.**) A charged particle with charge q moves on the z-axis with constant speed v, $\beta = v/c$. The electric and magnetic field generated by this charged particle is given by

$$\mathbf{E}(\mathbf{r},t) = (1-\beta^2)\frac{q}{4\pi\varepsilon_0}\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-vt)\hat{\mathbf{k}}}{[(x^2+y^2)(1-\beta^2)+(z-vt)^2]^{\frac{3}{2}}}, \tag{7a}$$

$$c\mathbf{B}(\mathbf{r},t) = \beta(1-\beta^2)\frac{q}{4\pi\varepsilon_0}\frac{-y\hat{\mathbf{i}} + x\hat{\mathbf{j}}}{[(x^2+y^2)(1-\beta^2)+(z-vt)^2]^{\frac{3}{2}}}. \tag{7b}$$

Evaluate the electromagnetic momentum density for this configuration by evaluating

$$\mathbf{G}(\mathbf{r},t) = \varepsilon_0\mathbf{E}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t). \tag{8}$$

5. (**20 points.**) For a particle moving very close to the speed of light, $\beta \to 1$, we can write the leading order contributions in $(1-\beta^2)$ for the electric and magnetic fields as

$$\mathbf{E}(\mathbf{r},t) = \begin{cases} \dfrac{1}{\sqrt{1-\beta^2}} \dfrac{q}{4\pi\varepsilon_0} \dfrac{\hat{\boldsymbol{\rho}}}{\rho^2}, & z = vt, \\ 0, & z \neq vt, \end{cases} \tag{9a}$$

$$c\mathbf{B}(\mathbf{r},t) = \begin{cases} \dfrac{\beta}{\sqrt{1-\beta^2}} \dfrac{q}{4\pi\varepsilon_0} \dfrac{\hat{\boldsymbol{\phi}}}{\rho^2}, & z = vt, \\ 0, & z \neq vt, \end{cases} \tag{9b}$$

where $\boldsymbol{\rho} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ and $\boldsymbol{\phi} = -y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$. These fields are confined on a plane perpendicular to direction of motion. The energy density of the electric field is given by

$$U_e(\mathbf{r},t) = \frac{\varepsilon_0}{2} E^2, \tag{10}$$

and the energy density of the magnetic field is given by

$$U_m(\mathbf{r},t) = \frac{1}{2\mu_0} B^2. \tag{11}$$

(a) Determine the ratio of electric to magnetic energy density,

$$\frac{U_m(\mathbf{r},t)}{U_e(\mathbf{r},t)}, \tag{12}$$

for the above configuration.

(b) Evaluate

$$\mathbf{E}(\mathbf{r},t) \cdot \mathbf{B}(\mathbf{r},t) \tag{13}$$

for the above configuration.

(c) A plane wave is characterized by $U_e = U_m$ and $\mathbf{E} \cdot \mathbf{B} = 0$. Does the above configuration satisfy the characteristics of a plane wave?