Homework No. 02 (Spring 2014)

PHYS 420: Electricity and Magnetism II

Due date: Friday, 2014 Feb 7, 4.30pm

1. Verify that the right hand side of

$$(-\mathbf{a}\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{\mathbf{a}\cdot\mathbf{r}}{r^3} \tag{1}$$

is a solution to Laplace's equation for $\mathbf{r}\neq 0$. Further, verify the relation

$$(\mathbf{a}_1\cdot\boldsymbol{\nabla})(\mathbf{a}_2\cdot\boldsymbol{\nabla})\frac{1}{r} = \frac{1}{r^5}\Big[3(\mathbf{a}_1\cdot\mathbf{r})(\mathbf{a}_2\cdot\mathbf{r}) - (\mathbf{a}_1\cdot\mathbf{a}_2)r^2\Big], \tag{2}$$

which is also a solution to Laplace's equation for $\mathbf{r}\neq 0$, but need not be verified here.

2. The generating function for the spherical harmonics, $Y_{lm}(\theta,\phi)$, is

$$\frac{1}{l!}\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right)^l = \sum_{m=-l}^{l}\sqrt{\frac{4\pi}{2l+1}}\,Y_{lm}(\theta,\phi)\,\psi_{lm}, \tag{3}$$

where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{4}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_-y_+), \tag{5}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}}\frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{6}$$

and

$$Y_{lm}(\theta,\phi) = e^{im\phi}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!}{(l-m)!}}\frac{1}{(\sin\theta)^m}\left(\frac{d}{d\cos\theta}\right)^{l-m}\frac{(\cos^2\theta-1)^l}{2^l l!}. \tag{7}$$

Show that

$$\left(\mathbf{a}\cdot\frac{\mathbf{r}}{r}\right) \tag{8}$$

is unchanged by the substitution: $y_+\leftrightarrow y_-$, $\theta\to-\theta$, $\phi\to-\phi$. Thus, show that

$$Y_{lm}(\theta,\phi) = Y_{l,-m}(-\theta,-\phi). \tag{9}$$

3. Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta, \phi)$ for $l = 0, 1, 2$, by completing the $l - m$ differentiations in Eq. (7). Use the result in Eq. (9) to reduce the work by about half.

4. Legendre polynomials of order l is given by (for $|t| < 1$)

$$P_l(t) = \left(\frac{d}{dt}\right)^l \frac{(t^2-1)^l}{2^l l!}. \tag{10}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the l differentiations in Eq. (10).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{11}$$

5. An integral representation for Legendre polynomial $P_l(\cos\theta)$ is

$$P_l(\cos\theta) = \int_0^\pi \frac{d\alpha}{\pi} \big[\cos\theta - i\sin\theta\cos\alpha\big]^l. \tag{12}$$

   (a) Use the integral representation for $J_0(t)$,

$$J_0(t) = \int_0^{2\pi} \frac{d\alpha}{2\pi}\, e^{it\cos\alpha}, \tag{13}$$

   to show that

$$P_l(\cos\theta) = \left(\cos\theta - \sin\theta\frac{d}{dt}\right)^l J_0(t)\bigg|_{t=0}. \tag{14}$$

   Verify this for $l = 0, 1, 2$.

   (b) Now let $\theta = x/l$ and, for fixed x, consider the limit $l \to \infty$, to obtain

$$\lim_{l\to\infty} P_l\left(\cos\frac{x}{l}\right) = J_0(x), \tag{15}$$

   which is often used in the approximate form

$$\theta \ll 1, l \gg 1: \qquad P_l(\cos\theta) \sim J_0(l\theta). \tag{16}$$

   (c) For what geometrical reason does one expect an asymptotic connection between spherical and cylindrical coordinate functions? (Hint: Green's function for planar geometry can be written in terms of J_m.)