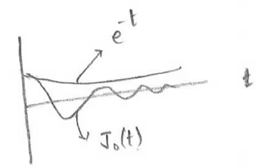
Set - 21

Modified Bessel functions (cylindrical geometry)

(1) The Green's function for planar geometry is suitably written in terms of Bessel function

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2}\, e^{i\vec{k}_\perp\cdot(\vec{r}-\vec{r}')_\perp}\, \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int_0^\infty dk_\perp\, J_0(k_\perp P)\, e^{-k_\perp|z-z'|} \qquad \vec{P} = (\vec{r}-\vec{r}')_\perp$$



where

$$J_0(t) = \int_0^{2\pi}\frac{d\alpha}{2\pi}\, e^{it\cos\alpha}$$

(2) Notice that

$$\nabla_\perp^2\, J_0(k_\perp P) = -k_\perp^2\, J_0(k_\perp P) \quad \rightarrow \text{oscillatory}$$

$$\frac{\partial^2}{\partial z^2}\, e^{-k_\perp|z-z'|} = +k_\perp^2\, e^{-k_\perp|z-z'|} \quad \rightarrow \text{exponentially decreasing}$$

(3) Observe the integral

$$\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,\frac{e^{ik_z(z-z')}}{k_z^2+k_\perp^2} = \frac{1}{2k_\perp}\, e^{-k_\perp|z-z'|}$$

which can be evaluated using contour integrals.
Is there an elementary way to derive this?

4 Using 3 in 1

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0} \, 2\int_0^\infty k_\perp dk_\perp \, J_0(k_\perp P) \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} \frac{e^{ik_z(z-z')}}{k_z^2 + k_\perp^2}$$

$$= \frac{1}{2\pi\epsilon_0} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} \underbrace{\boxed{\int_0^\infty k_\perp dk_\perp \frac{J_0(k_\perp P)}{k_\perp^2 + k_z^2}}}_{K_0(k_z P)}$$

5 Modified Bessel functions are defined as

$$K_0(t) = \int_0^\infty s\,ds \, \frac{J_0(s)}{s^2 + t^2} \qquad 0 < t < \infty$$

6 Notice :

$e^{ik_z(z-z')} \longrightarrow$ oscillatory

$K_0(k_\perp P) \longrightarrow$ exponentially decreasing.

7 Thus, the suitable forms are.

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi\epsilon_0} \int_0^\infty dk_\perp \, J_0(k_\perp P) \, e^{-k_\perp|z-z'|} \longrightarrow \text{planar geometry}$$

$$= \frac{1}{2\pi\epsilon_0} \int_{-\infty}^{+\infty} \frac{dk_z}{2\pi} e^{ik_z(z-z')} K_0(k_z P) \longrightarrow \text{cylindrical geometry}$$

8 Notice

$$G(\vec{r},\vec{r}\,') = \frac{1}{2\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z\rho)$$

$$= \frac{1}{2\pi\epsilon_0}\int_{-\infty}^{0}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z\rho) + \frac{1}{2\pi\epsilon_0}\int_{0}^{\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z\rho)$$

$$k_z = -k_z'$$

$$= \frac{1}{2\pi\epsilon_0}\int_{\infty}^{0}(-1)\frac{dk_z'}{2\pi}\, e^{-ik_z'(z-z')}\, K_0(-k_z'\rho) + \frac{1}{2\pi\epsilon_0}\int_{0}^{\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\, K_0(k_z\rho)$$

9 Using $K_0(-t) = K_0(t)$, see 5, we have.

$$G(\vec{r},\vec{r}\,') = \frac{1}{2\pi\epsilon_0}\int_{0}^{\infty}\frac{dk_z}{2\pi}\left[e^{ik_z(z-z')} + e^{-ik_z(z-z')}\right] K_0(k_z\rho)$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{2}{\pi}\int_{0}^{\infty}dk_z\, \cos\left(k_z(z-z')\right)\, K_0(k_z\rho)$$

which brings out the oscillatory nature in the z-direction more explicitly.

(10) To find out the differential equation satisfied by the modified Bessel function we write

$$K_0(k_z \rho) = \int_0^\infty k_\perp \, dk_\perp \, \frac{J_0(k_\perp \rho)}{k_\perp^2 + k_z^2}$$

$$= \int_0^\infty k_\perp \, dk_\perp \int_0^{2\pi} \frac{d\alpha}{2\pi} \, e^{i k_\perp \rho \cos\alpha} \, \frac{1}{k_\perp^2 + k_z^2}$$

$$= 2\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \, e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \, \frac{1}{k_\perp^2 + k_z^2}$$

(11) The above form suggests

$$(-\nabla_\perp^2 + k_z^2) \, K_0(k_z \rho) = 2\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \left[-(i k_\perp)^2 + k_z^2 \right] \frac{e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp}}{k_\perp^2 + k_z^2}$$

$$= 2\pi \int \frac{d^2 k_\perp}{(2\pi)^2} \, e^{i \vec{k}_\perp \cdot (\vec{r} - \vec{r}')}$$

$$= 2\pi \, \delta^{(2)}(\vec{r}_\perp - \vec{r}'_\perp)$$

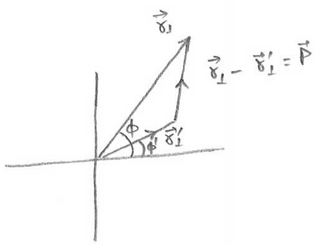
Thus, we have the differential equation

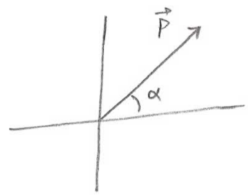
$$(-\nabla_\perp^2 + k_z^2) \, \frac{1}{2\pi} K_0(k_z \rho) = \delta^{(2)}(\vec{r}_\perp - \vec{r}'_\perp)$$

(12) Let us now argue that

$$\vec{P} = \vec{r}_\perp - \vec{r}_\perp'$$

$$\delta^{(2)}(\vec{r}_\perp - \vec{r}_\perp') = \frac{\delta(P)}{P}\frac{1}{2\pi}$$





$$\vec{r}_\perp = (\rho \cos\phi, \rho \sin\phi)$$

$$\vec{r}_\perp' = (\rho' \cos\phi', \rho' \sin\phi')$$

$$\vec{P} = (\rho \cos\phi - \rho' \cos\phi', \rho \sin\phi - \rho' \sin\phi')$$

$$P = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi')}$$

Since

$$\vec{P} = P\,\hat{P} + 0\,\hat{\alpha}$$

we can write

$$\delta^{(2)}(\vec{P}) = \frac{\delta(P)}{P}\,\delta(\alpha)$$

In the limit $P \to 0$ we have.

$$\delta^{(2)}(\vec{P}) = \frac{\delta(P)}{P}\frac{1}{2\pi}$$

because we require.

$$\int_0^\infty P\,dP \int_0^{2\pi} d\alpha\ \delta^{(2)}(\vec{P}) = 1.$$

13 Using (12) in (11) we have.

$$\left(-\nabla_\perp^2 + k_z^2\right) \frac{1}{2\pi} K_0(k_z \rho) = \frac{\delta(\rho)}{\rho} \frac{1}{2\pi}$$

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} + k_z^2\right] K_0(k_z \rho) = \frac{\delta(\rho)}{\rho}$$

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] K_0(t) = \frac{\delta(t)}{t}$$

14 For $t \neq 0$, replacing $t \to it$ we obtain

$$\left[\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] K_0(it) = 0$$

which in comparison to the Bessel equation

$$\left[\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + 1\right] J_0(t) = 0$$

reveals the connection

$$K_0(it) = J_0(t).$$

Thus, the name modified Bessel function.

(15) Another way to verify consistency is

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+k_z^2\right]K_0(k_z\rho) = \left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+k_z^2\right]\int_0^\infty k_\perp dk_\perp \frac{J_0(k_\perp\rho)}{k_\perp^2+k_z^2}$$

$$= \int_0^\infty k_\perp dk_\perp \frac{1}{k_\perp^2+k_z^2}\left[-\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho}+k_z^2\right]J_0(k_\perp\rho)$$

$$= \int_0^\infty k_\perp dk_\perp \frac{1}{k_\perp^2+k_z^2}\left[-(-k_\perp^2)+k_z^2\right]J_0(k_\perp\rho)$$

$$= \int_0^\infty k_\perp dk_\perp J_0(k_\perp\rho)$$

$$= \frac{\delta(\rho)}{\rho} \quad \text{using} \quad \int_0^\infty k_\perp dk_\perp J_0(k_\perp\rho)J_0(k_\perp\rho') = \frac{\delta(\rho-\rho')}{\rho}$$

(16)
$$\left(-\nabla_\perp^2+k_z^2\right)K_0(k_z\rho) = \frac{\delta(\rho)}{\rho}$$

Using divergence theorem in two dimensions.

$$-\int d^2\rho\, \nabla_\perp^2 K_0(k_z\rho) + k_z^2\int d^2\rho\, K_0(k_z\rho) = \int d^2\rho\,\frac{\delta(\rho)}{\rho}$$

$$-\int d\vec{a}\cdot\vec{\nabla}_\perp K_0(k_z\rho) + k_z^2\int d^2\rho\, K_0(k_z\rho) = 2\pi$$

evaluate on a circle

17 Evaluating 16 on a circle of infinitely small radius we have

$$d\vec{a} \cdot \vec{\nabla}_\perp = dl \; \hat{\rho} \cdot \hat{\rho} \frac{d}{d\rho}$$

$$- \oint_{\substack{\text{circle of} \\ \text{radius } \rho}} dl \; \frac{d}{d\rho} K_0(k_z \rho) + 0 = 2\pi$$

$$- 2\pi \rho \frac{d}{d\rho} K_0(k_z \rho) = 2\pi$$

$$- t \frac{d}{dt} K_0(t) = 1$$

$$\frac{d}{dt} K_0(t) = - \frac{1}{t}$$

$$\Rightarrow \quad K_0(t) = - \ln t + \text{constant}$$

Actually, exact form is

$$K_0(t) = \ln \frac{2}{t} - \gamma$$

γ = Euler's constant $= 0.577\ldots$