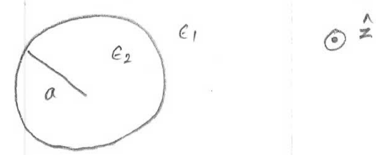
Free Green's function - cylindrical geometry

① In preparation towards finding Green's function for a dielectric cylinder

$$\vec{r} = (\rho, \phi, z)$$



we consider

$$-\vec{\nabla}\cdot\epsilon(\rho)\,\vec{\nabla}\,G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

② Let $\epsilon(\rho) = \epsilon_0$

$$-\epsilon_0\,\nabla^2 G(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,')$$

③ $$G(\vec{r},\vec{r}\,') = \int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,e^{ik_z(z-z')}\,\frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\,g_m(\rho,\rho';k_z)$$

$$\delta^{(3)}(\vec{r}-\vec{r}\,') = \delta(z-z')\,\delta(\phi-\phi')\,\frac{\delta(\rho-\rho')}{\rho}$$

$$= \int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,e^{ik_z(z-z')}\,\frac{1}{2\pi}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\phi')}\,\frac{\delta(\rho-\rho')}{\rho}$$

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}.$$

4 Thus, we have

$$-\epsilon_0 \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{(im)^2}{\rho^2} + (ik_z)^2 \right] g_m(\rho,\rho';k_z) = \frac{\delta(\rho-\rho')}{\rho}.$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2} + k_z^2 \right] g_m(\rho,\rho';k_z) = \frac{1}{\epsilon_0}\frac{\delta(\rho-\rho')}{\rho}.$$

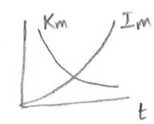
5 For $\rho = \rho'$ we can write

$$g_m(\rho,\rho';k_z) = \begin{cases} A\, I_m(k_z\rho) + B\, K_m(k_z\rho) & 0 \le \rho < \rho' \\ C\, I_m(k_z\rho) + D\, K_m(k_z\rho) & 0 \le \rho' < \rho \end{cases}$$

6 Requiring $g_m(\pm\infty, \rho'; k_z) = 0$ we immediately have $B = 0$ and $C = 0$.

$I_m \rightarrow e^{\kappa x}$

$K_m \rightarrow e^{-\kappa x}$



7 Integrating around $\rho = \rho'$ we obtain the boundary conditions

$$-\rho\frac{\partial}{\partial\rho} g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = \frac{1}{\epsilon_0}$$

and

$$g_m(\rho,\rho';k_z)\Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0$$

(8) Using (5), (6) and (7) we have.

$$D \, K_m(k_z \rho') - A \, I_m(k_z \rho') = 0$$

$$D \frac{d}{d\rho'} K_m(k_z \rho') - A \frac{d}{d\rho'} I_m(k_z \rho') = -\frac{1}{\epsilon_0 \rho'}$$

(9)
$$A = -\frac{1}{\epsilon_0 \rho'} \frac{K_m(k_z \rho')}{\left[I_m(k_z \rho') \frac{d}{d\rho'} K_m(k_z \rho') - K_m(k_z \rho') \frac{d}{d\rho'} I_m(k_z \rho') \right]}$$

$$D = -\frac{1}{\epsilon_0 \rho'} \frac{I_m(k_z \rho')}{\left[I_m(k_z \rho') \frac{d}{d\rho'} K_m(k_z \rho') - K_m(k_z \rho') \frac{d}{d\rho'} I_m(k_z \rho') \right]}$$

(10) We notice the appearance of the Wronskian.

$$K_m \times \quad \left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2} + k_z^2 \right] I_m(k_z \rho) = 0$$

$$I_m \times \quad \left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{m^2}{\rho^2} + k_z^2 \right] K_m(k_z \rho) = 0$$

Thus,

$$I_m(k_z \rho) \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} K_m(k_z \rho) - K_m(k_z \rho) \frac{1}{\rho} \frac{d}{d\rho} \rho \frac{d}{d\rho} I_m(k_z \rho) = 0$$

(11) Let $k_\perp \rho = t$

$$I_m \frac{d}{dt} t \frac{d}{dt} K_m - K_m \frac{d}{dt} t \frac{d}{dt} I_m = 0$$

$$I_m' = \frac{d}{dt} I_m$$

$$I_m \frac{d}{dt}\left(t K_m'\right) - K_m \frac{d}{dt}\left(t I_m'\right) = 0$$

$$K_m' = \frac{d}{dt} K_m$$

$$\frac{d}{dt}\left[I_m t K_m' - K_m t I_m' \right] = 0$$

c - constant.

$$\Rightarrow \quad I_m K_m' - K_m I_m' = \frac{c}{t}$$

(12) For $t \gg 1$ we have.

$$I_m(t) \sim \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{t}} e^{t}$$

$$K_m(t) \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{t}} e^{-t}.$$

$$I_m K_m' - K_m I_m' = \frac{1}{2} \frac{1}{\sqrt{t}} e^{t} \frac{d}{dt}\left(\frac{1}{\sqrt{t}} e^{-t}\right) - \frac{1}{2} \frac{1}{\sqrt{2}} e^{-t} \frac{d}{dt}\left(\frac{1}{\sqrt{t}} e^{t}\right)$$

$$= -\frac{1}{t}$$

$$\Rightarrow \quad c = -1$$

Thus,

$$I_m K_m' - K_m I_m' = -\frac{1}{t}.$$

13 Using 12 in 9

$$A = \frac{1}{\epsilon_0} K_m(k_z \rho')$$

$$D = \frac{1}{\epsilon_0} I_m(k_z \rho')$$

14 Using 13 in 5

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\epsilon_0} K_m(k_z \rho') \, I_m(k_z \rho) & 0 \leq \rho < \rho' \\ \frac{1}{\epsilon_0} I_m(k_z \rho') \, K_m(k_z \rho) & 0 \leq \rho' < \rho \end{cases}$$

$$= \frac{1}{\epsilon_0} I_m(k_z \rho_<) \, K_m(k_z \rho_>)$$

where

$$\rho_< = \text{Minimum}(\rho, \rho')$$

$$\rho_> = \text{Maximum}(\rho, \rho')$$