SET-24

Interaction energies

1. For two charges we have

q_1 • • q_2

$$E = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

2. For three charge we have.

$$E = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{q_2 q_3}{|\vec{r}_2 - \vec{r}_3|} + \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|}\right]$$

$$= \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

$i \neq j$: removes the self interaction terms

$\frac{1}{2}$: removes the double counting

3. For a continuous charge distribution the above generalizes to

$$E = \frac{1}{2} \int d^3r \int d^3r' \frac{\rho(\vec{r})\,\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} - E_{self}$$

$$= \frac{1}{2} \int d^3r \int d^3r' \,\rho(\vec{r})\, G_0(\vec{r} - \vec{r}')\, \rho(\vec{r}') - E_{self},$$

where E_{self} is the self energy, which does not contribute to physical observables.

4 For a given charge distribution in vacuum we have

$$\vec{\nabla} \cdot \epsilon_0 \vec{E}(\vec{r}) = \rho(\vec{r})$$

$$-(\vec{\nabla} \cdot \epsilon_0 \vec{\nabla}) G_0 = 1$$

$$G_0 = (-\vec{\nabla} \cdot \epsilon_0 \vec{\nabla})^{-1}$$

$$-(\vec{\nabla} \cdot \epsilon_0 \vec{\nabla}) \phi(\vec{r}) = \rho(\vec{r})$$

$$\phi(\vec{r}) = \int d^3r' \, G_0(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

$$E = \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r}) \, G_0(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

5 For a given charge distribution in the presence of a dielectric material we have

$$\vec{\nabla} \cdot \epsilon_0 \vec{E} = \underbrace{\rho}_{\text{charge}} - \underbrace{\vec{\nabla} \cdot \vec{P}}_{\text{effective charge due to material}}$$

$$\vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho$$

$$\epsilon \vec{E} = \epsilon_0 \vec{E} + \vec{P}$$

$$\vec{\nabla} \cdot \epsilon \vec{E} = \rho$$

$$-(\vec{\nabla} \cdot \epsilon \vec{\nabla}) G = 1$$

$$G = -(\vec{\nabla} \cdot \epsilon \vec{\nabla})^{-1}$$

$$-(\vec{\nabla} \cdot \epsilon \vec{\nabla}) \phi(\vec{r}) = \rho(\vec{r})$$

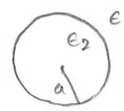
$$\phi(\vec{r}) = \int d^3r' \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

$$E = \frac{1}{2} \int d^3r \int d^3r' \, \rho(\vec{r}) \, G(\vec{r}, \vec{r}') \, \rho(\vec{r}')$$

(6) We have found the Green's funct for planar and cylindrical geometries.

Planar : $\epsilon_2 \,|\, \epsilon_1$ $\quad (z'<a)$

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_x}{2\pi}e^{ik_x(x-x')}\int_{-\infty}^{+\infty}\frac{dk_y}{2\pi}e^{ik_y(y-y')}\left[\underbrace{\frac{1}{2k_\perp}e^{-k_\perp|z-z'|}}_{G_0} - \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{1}{2k_\perp}e^{-k_\perp|z-a|}e^{-k_\perp|z'-a|}\right]$$

Cylinder :  $\epsilon_1 \quad \epsilon_1\to\infty$

$$G(\vec{r},\vec{r}') = \frac{1}{\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}e^{ik_z(z-z')}\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}e^{im(\phi-\phi')}\left[\underbrace{I_m(k_z\rho_<)K_m(k_z\rho_>)}_{G_0} - I_m(k_z\rho)I_m(k_z\rho')\frac{K_a}{I_a}\right]$$

(7) Interaction energy between charge density and dielectric material is given by

$$E_{int} = \frac{1}{2}\int d^3r\int d^3r'\,\rho(\vec{r})\left[G(\vec{r},\vec{r}') - G_0(\vec{r},\vec{r}')\right]\rho(\vec{r}'),$$

which is the change in energy of the charge configuration due to the introduction of the dielectric material.

(8) For a point charge we have

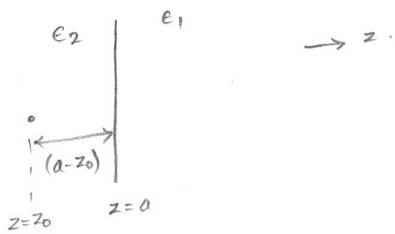
$$\rho(\vec{r}) = q\, \delta^{(3)}(\vec{r} - \vec{r}_0)$$

$\vec{r}_0 \rightarrow$ position of charge

which leads to the interaction energy of a point charge with a dielectric material to be

$$E_{int} = \frac{q^2}{2}\left[G(\vec{r}, \vec{r}') - G_0(\vec{r}, \vec{r}') \right]_{\vec{r} = \vec{r}' = \vec{r}_0}$$

(9) Using (6) in (8) we have the interaction energy for a charge and a planar dielectric to be



$$E_{int} = \frac{q^2}{2} \frac{(-1)}{\epsilon_2} \int \frac{d^2 k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp \cdot (\vec{r} - \vec{r}')_\perp} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{2k_\perp} e^{-k_\perp |z-a|} e^{-k_\perp |z'-a|} \Bigg|_{\vec{r} = \vec{r}' = \vec{r}_0}$$

$$= -\frac{q^2}{4\pi\epsilon_2} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{2\pi} \int d^2 k_\perp \frac{1}{2k_\perp} e^{-2k_\perp |z_0 - a|}$$

$$= -\frac{q^2}{4\pi\epsilon_2} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{4|a - z_0|}$$

(10) Force acting on the charge is given by

$$F = -\frac{\partial}{\partial z_0} E_{int}$$

$$= \frac{q^2}{4\pi\epsilon_2} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{\partial}{\partial z_0} \frac{1}{4|a - z_0|}$$

for $z_0 < a$.

$$= \frac{q^2}{4\pi\epsilon_2} \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \frac{1}{4|a - z_0|^2}$$

(11) Comparing the above expression for the force between two oppositely charged

$$F = -\frac{1}{4\pi\epsilon_2} \frac{q\, q_{image}}{r^2}$$

we can interpret

$$q_{image} = -q \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}.$$

$$r = 2|a - z_0|$$

(12) Perfect conductor limit : $(\epsilon_1 \to \infty)$

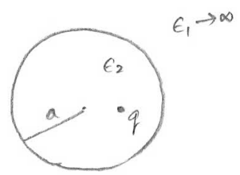
$$E_{int} = -\frac{q^2}{4\pi\epsilon_2} \frac{1}{4|a - z_0|}$$

$$F = \frac{q^2}{4\pi\epsilon_2} \frac{1}{4|a - z_0|^2}$$

$$q_{image} = -q$$

$$r = 2|a - z_0|$$

(13) Let us next consider a charge inside a perfectly conducting cylinder



$$\varrho(\vec{r}) = q\, \delta^{(3)}(\vec{r}-\vec{r}_0)$$

$$\vec{r}_0 = (\varrho_0, \phi_0, z_0)$$

(14) $$E_{int} = \frac{q^2}{2}\Big[\, G(\vec{r},\vec{r}\,') - G_0(\vec{r},\vec{r}\,')\Big]_{\vec{r}=\vec{r}\,'=\vec{r}_0}$$

Using cylinder Green's function from (6)

$$E_{int} = -\frac{q^2}{2\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\, e^{ik_z(z-z')}\sum_{m=-\infty}^{+\infty}\frac{1}{2\pi}\, e^{im(\phi-\phi')}\, I_m(k_z\varrho)\, I_m(k_z\varrho')\,\frac{K_a}{I_a}\Bigg|_{\vec{r}=\vec{r}\,'=\vec{r}_0}$$

$$= -\frac{q^2}{4\pi\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\sum_{m=-\infty}^{+\infty}\Big[I_m(k_z\varrho_0)\Big]^2\frac{K_m(k_z a)}{I_m(k_z a)}$$

(15) Let the charge be on the axis,

$\varrho_0 = 0$,

$$E_{int} = -\frac{q^2}{4\pi\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\sum_{m=-\infty}^{+\infty}\Big[I_m(0)\Big]^2\frac{K_m(k_z a)}{I_m(k_z a)}$$

$$I_m(0) = \begin{cases} 1 & \text{if } m=0 \\ 0 & \text{if } m>0 \end{cases}$$

$$= -\frac{q^2}{4\pi\epsilon_2}\int_{-\infty}^{+\infty}\frac{dk_z}{2\pi}\,\frac{K_0(k_z a)}{I_0(k_z a)}$$

16 Thus,

$$E_{int} = -\frac{q^2}{4\pi\epsilon_2}\frac{1}{2\pi a}\int_{-\infty}^{+\infty} dt\,\frac{K_0(t)}{I_0(t)}$$

$$= -\frac{q^2}{4\pi\epsilon_2}\frac{1}{\pi a}\int_{0}^{\infty} dt\,\frac{K_0(t)}{I_0(t)}$$

17 We can numerically evaluate, for example using Mathematica,

$$\int_{0}^{\infty} dt\,\frac{K_0(t)}{I_0(t)} = 1.36768\ldots$$

18 Using 17 in 16 we have

$$E_{int} = -\frac{q^2}{4\pi\epsilon_2}\frac{1.36768\ldots}{\pi a}.$$