Final Exam No. (Spring 2014)

PHYS 530A: Quantum Mechanics II

Date: 2014 May 8

1. (**20 points.**) A 4×4 matrix A satisfies the equation

$$A^4 = 1. \tag{1}$$

Given that the eigenvalues of A are non-degenerate, find all the eigenvalues of A.

2. (**20 points.**) The components of the position and momentum operator, $\mathbf{r}$ and $\mathbf{p}$, respectively, satisfy the commutation relations $[r_i, p_j] = i\hbar\delta_{ij}$. Evaluate the following:

   (a) $\mathbf{r} \cdot \mathbf{p} - \mathbf{p} \cdot \mathbf{r} =$

   (b) $\mathbf{r} \times \mathbf{p} + \mathbf{p} \times \mathbf{r} =$

3. (**20 points.**) Evaluate the commutation relation between $1/r$, the inverse of the magnitude of the position operator $\mathbf{r}$, and the angular momentum operator $\mathbf{L}$,

$$\left[\frac{1}{r}, \mathbf{L}\right], \tag{2}$$

which is encountered, for example, in the analysis of hydrogen atom.

4. (**20 points.**) Consider the total angular momentum states $|j, m\rangle$ for the composite system built out of two individual angular momenta $j_1 = 4$ and $j_2 = 1$.

   (a) Determine the total number of states by counting the individual states,

$$\left(\sum_{m_1=-j_1}^{j_1}\right)\left(\sum_{m_2=-j_2}^{j_2}\right). \tag{3}$$

   Repeat this by counting the number of total angular momentum states,

$$\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j}. \tag{4}$$

   (b) Construct the total angular momentum state $|4, 4\rangle$.

5. (**10 points.**) Verify that the axial vector

$$\mathbf{A} = \frac{\mathbf{r}}{r} - \frac{1}{\mu Z e^2}\frac{1}{2}(\mathbf{p}\times\mathbf{L} - \mathbf{L}\times\mathbf{p}), \tag{5}$$

which is a conserved quantity for hydrogenic atoms, satisfies

$$\mathbf{A}\cdot\mathbf{L} = 0 \qquad \text{and} \qquad \mathbf{L}\cdot\mathbf{A} = 0. \tag{6}$$

Verify this in classical mechanics and in quantum mechanics, where physical variables are operators.

6. (**10 points.**) The transformation function relating the angular momentum eigenvectors between two coordinate frames, related by rotations described using Euler angles (ψ, θ, ϕ), is

$$\langle j, m|U(\psi,\theta,\phi)|j', m'\rangle = \delta_{jj'} e^{im\psi} U^{(j)}_{m,m'}(\theta) e^{im'\phi}, \tag{7}$$

where $U^{(j)}_{m,m'}(\theta)$ are generated by the relation

$$\frac{\bar{y}_+^{j+m}}{\sqrt{(j+m)!}}\frac{\bar{y}_-^{j-m}}{\sqrt{(j-m)!}} = \sum_{m'=-j}^{j} e^{im\psi} U^{(j)}_{m,m'}(\theta) e^{im'\phi} \frac{y_+^{j+m'}}{\sqrt{(j+m')!}}\frac{y_-^{j-m'}}{\sqrt{(j-m')!}}, \tag{8}$$

where

$$\begin{bmatrix} \bar{y}_+ \\ \bar{y}_- \end{bmatrix} = \begin{bmatrix} e^{i\frac{\psi}{2}}\cos\frac{\theta}{2}e^{i\frac{\phi}{2}} & e^{i\frac{\psi}{2}}\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ -e^{-i\frac{\psi}{2}}\sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & e^{-i\frac{\psi}{2}}\cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \end{bmatrix} \begin{bmatrix} y_+ \\ y_- \end{bmatrix}. \tag{9}$$

The above transformation function gives the probability amplitude relating measurements of angular momentum, or magnetic dipole moment, in two different directions related by the Euler angles.

The following experiment is performed using beams consisting of angular momentum $j = 2$. An initial Stern-Gerlach measurement selects the $j = 2$, $m = 2$, beam. A second measurement of $j = 2$, $m' = 1$, is made on this beam in a direction differing by angle θ. The probability relating these measurements is determined by

$$p(m, m'; \theta) = \left|\langle j, m|U(\psi,\theta,\phi)|j', m'\rangle\right|^2. \tag{10}$$

Extract the probability, $p(m = 2, m' = 1; \theta)$ for $j = 2$.