Homework No. 02 (Spring 2014)

PHYS 530A: Quantum Mechanics II

Due date: Wednesday, 2014 Feb 5, 4.30pm

1. Show that the commutator of two matrices,

$$\big[\mathbf{A}, \mathbf{B}\big] \equiv \mathbf{AB} - \mathbf{BA}, \tag{1}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big] = -\big[\mathbf{B}, \mathbf{A}\big]. \tag{2}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big] = a\big[\mathbf{A}, \mathbf{C}\big] + b\big[\mathbf{B}, \mathbf{C}\big]. \tag{3}$$

Further show that

$$\big[\mathbf{AB}, \mathbf{C}\big] = \mathbf{A}\big[\mathbf{B}, \mathbf{C}\big] + \big[\mathbf{A}, \mathbf{C}\big]\mathbf{B}. \tag{4}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]\big] + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]\big] + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]\big] = 0. \tag{5}$$

2. Show that the vector product of two vectors,

$$\big[\mathbf{A}, \mathbf{B}\big] \equiv \mathbf{A} \times \mathbf{B}, \tag{6}$$

satisfies the conditions for a Lie algebra, as does the Poisson bracket. In particular show that

(a) Antisymmetry:

$$\big[\mathbf{A}, \mathbf{B}\big] = -\big[\mathbf{B}, \mathbf{A}\big]. \tag{7}$$

(b) Bilinearity: (a and b are numbers.)

$$\big[a\mathbf{A} + b\mathbf{B}, \mathbf{C}\big] = a\big[\mathbf{A}, \mathbf{C}\big] + b\big[\mathbf{B}, \mathbf{C}\big]. \tag{8}$$

Further show that

$$\big[\mathbf{A} \times \mathbf{B}, \mathbf{C}\big] = \mathbf{A} \times \big[\mathbf{B}, \mathbf{C}\big] + \big[\mathbf{A}, \mathbf{C}\big] \times \mathbf{B}. \tag{9}$$

(c) Jacobi's identity:

$$\big[\mathbf{A}, \big[\mathbf{B}, \mathbf{C}\big]\big] + \big[\mathbf{B}, \big[\mathbf{C}, \mathbf{A}\big]\big] + \big[\mathbf{C}, \big[\mathbf{A}, \mathbf{B}\big]\big] = 0. \tag{10}$$

3. Given F and G are constants of motion, that is

$$\big[F, H\big] = 0 \qquad \text{and} \qquad \big[G, H\big] = 0. \tag{11}$$

Then, using Jacobi's identity, show that $\big[F, G\big]$ is also a constant of motion. Thus, conclude the following:

(a) If L_x and L_y are constants of motion, then L_z is also a constant of motion.

(b) If p_x and L_z are constants of motion, then p_y is also a constant of motion.

4. (Refer Goldstein, Sec. 9.5.) Hamiltonian for the motion of a ball (along the radial direction) near the surface of Earth is given by

$$H(z, p_z) = \frac{p_z^2}{2m} - mgz. \tag{12}$$

(a) Determine the equations of motions using

$$\frac{dz}{dt} = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \frac{dp_z}{dt} = -\frac{\partial H}{\partial z}. \tag{13}$$

Then, solve the coupled differential equations to find the familiar elementary solution

$$z = z_0 + \frac{p_0}{m} t + \frac{1}{2} g t^2. \tag{14}$$

(b) Next, determine the equations of motion using

$$\big[z, H\big] = \frac{\partial H}{\partial p_z} \qquad \text{and} \qquad \big[p_z, H\big] = -\frac{\partial H}{\partial z}. \tag{15}$$

Then, using

$$z = z_0 + t\big[z, H\big]_0 + \frac{1}{2} t^2 \big[\big[z, H\big], H\big]_0 + \cdots \tag{16}$$

rederive the elementary solution. Here the subscript zero refers to the initial conditions at $t = 0$.