Stationary action principle (Hamilton's principle)

(1) Hamilton's principle states that for any system there exists a function $L(q_i, \dot{q}_i, t)$, the Lagrangian, such that the integral

$$\int_{t_1}^{t_2} dt \; L(q_i, \dot{q}_i, t)$$

is an extremum for the true path.

(2) Let us begin by considering the case when L is independent of time and

$$\delta q(t_1) = \delta q(t_2) = 0 .$$

(3)
$$\delta \int_{t_1}^{t_2} dt \; L(q_i, \dot{q}_i) = \sum_i \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right)$$

Using $\delta \dot{q}_i = \frac{d}{dt} \delta q_i$ and integrating by parts,

$$\delta \int_{t_1}^{t_2} dt \; L = \sum_i \int_{t_1}^{t_2} dt \; \delta q_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) + \underbrace{\sum_i \int_{t_1}^{t_2} dt \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]}_{=0}$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \qquad i = 1, 2, \ldots, N.$$

4 Example: <u>Non-relativistic charged particle</u>

Consider the motion of a charged particle in a known electromagnetic field.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$

$$\frac{d}{dt}(m\vec{v}) = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

$$\frac{d}{dt}(m\vec{v}) = q\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\vec{v}\times(\vec{\nabla}\times\vec{A})\right]$$

$$= q\left[-\vec{\nabla}\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \frac{1}{c}\vec{\nabla}(\vec{v}\cdot\vec{A}) - \frac{1}{c}\vec{v}\cdot\vec{\nabla}\vec{A}\right]$$

$\vec{v}$ is position independent

$$(\vec{\nabla}\vec{A})\cdot\vec{v} = \vec{\nabla}(\vec{v}\cdot\vec{A})$$

$$= q\left[-\vec{\nabla}\phi + \frac{1}{c}\vec{\nabla}(\vec{v}\cdot\vec{A}) - \frac{1}{c}\left\{\frac{\partial \vec{A}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{A}\right\}\right]$$

$$= q\left[-\vec{\nabla}\phi + \frac{1}{c}\vec{\nabla}(\vec{v}\cdot\vec{A}) - \frac{1}{c}\frac{d\vec{A}}{dt}\right]$$

$$\frac{d}{dt}\left(m\vec{v} + \frac{q}{c}\vec{A}\right) = \vec{\nabla}\left[-q\phi + \frac{q}{c}\vec{v}\cdot\vec{A}\right] \qquad \text{—(i)}$$

5 We need to compare the above with, see 3,

$$\frac{d}{dt}\frac{\partial L}{\partial \vec{v}} = \vec{\nabla} L$$

6 Comparing 4-(i) and 5 we have

$$\frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c}\vec{A} \qquad \text{— (i)}$$

and

$$L = -q\phi + \frac{q}{c}\vec{v}\cdot\vec{A} + (\text{function independent of } \vec{x}) \qquad \text{— (ii)}$$

7 Eq. 6 -(i) implies

$$L = \frac{1}{2}mv^2 + \frac{q}{c}\vec{v}\cdot\vec{A} + (\text{function independent of } \vec{v})$$

8 Together, 6 and 7 is consistent with the solution

$$L = \frac{1}{2}mv^2 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$$

9 For the relativistic case we will have. ← HW

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

find the Lagrangian.

10 Hamiltonian form of the equations of motion are got by the definitions

define : $p_i = \frac{\partial L}{\partial \dot{q}_i}$ $\rightarrow$ canonical momentum

define : $H = \sum_i p_i \dot{q}_i - L = H(p_i, q_i)$ $\rightarrow$ Hamiltonian

11 Note that

$$dH = \sum_i \left(dp_i \, \dot{q}_i + p_i \, d\dot{q}_i \right) - \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \overbrace{\frac{\partial L}{\partial \dot{q}_i}}^{p_i} d\dot{q}_i \right)$$

$$= \sum_i \left(\dot{q}_i \, dp_i - \frac{\partial L}{\partial q_i} dq_i \right)$$

Thus, $H = H(p_i, q_i)$ is justified.

12 Comparing 11 with

$$dH = \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

we obtain ~~the~~ Hamilton's equations of motion

$$\dot{p}_i = - \frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

(13) If H is independent of time

$$H = H(p_i, q_i)$$

and

$$\frac{dH}{dt} = \sum_i \left(\frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i \right)$$

$$= \sum_i \left(\dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i \right)$$ (using Hamilton's equations of motion.)

$$= 0$$

which is the statement of conservation of energy.

(13) If H is time dependent

$$H = H(p_i, q_i, t)$$

and

$$dH = dt \frac{\partial H}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right)$$

which implies ← HW.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

(14) Hamiltonian for the example of a non-relativistic charged particle can be found now.

$$L = \frac{1}{2} m v^2 + \frac{q}{c} \vec{v} \cdot \vec{A} - q\phi$$

(see (8))

Define :

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{q}{c} \vec{A}$$

Define

$$\begin{aligned} H(\vec{p}, \vec{x}) &= \vec{p} \cdot \vec{v} - L \\ &= \vec{p} \cdot \vec{v} - \frac{1}{2} m v^2 - \frac{q}{c} \vec{v} \cdot \vec{A} + q\phi \\ &= \left(m\vec{v} + \frac{q}{c} \vec{A} \right) \cdot \vec{v} - \frac{1}{2} m v^2 - \frac{q}{c} \vec{v} \cdot \vec{A} + q\phi \\ &= \frac{1}{2} m v^2 + q\phi \\ &= \frac{1}{2m} \left(\vec{p} - \frac{q}{c} \vec{A} \right)^2 + q\phi \end{aligned}$$

(15) The Hamiltonian for the relativistic case will be covered in homework. $\leftarrow$ HW

(16) In general we will have.

$$L = L\left(x(t) \,,\, \frac{d}{dt}x(t) \,,\, t \right)$$

implicit explicit

$$L = L\left(x, \frac{dx}{dt}, t \right)$$ — display only the explicit time dependence.

(17) Construct the action

$$W[\,x\,;\,t_1, t_2\,] = \int_{t_1}^{t_2} dt\; L\left(x, \frac{dx}{dt}, t \right)$$

$$W + \delta W = \int_{t_1+\delta t_1}^{t_2+\delta t_2} dt\; L\left(x+\delta x \,,\, \frac{d}{dt}(x+\delta x) \,,\, t \right)$$

(18) Substitute

$$t' = t - \delta t\,(t)$$

with the requirement

$$\delta t\,(t_1) = \delta t_1$$

$$\delta t\,(t_2) = \delta t_2 .$$

(19)

$$t_1' = (t_1 + \delta t_1) - \delta t(t_1 + \delta t_1)$$

$$= t_1 + \delta t_1 - \delta t(t_1) + O(\delta t)^2$$

$$= t_1 + \delta t_1 - \delta t_1$$

$$= t_1$$

$$t_2' = (t_2 + \delta t_2) - \delta t(t_2 + \delta t_2)$$

$$= t_2 + \delta t_2 - \delta t(t_2) + O(\delta t)^2$$

$$= t_2 + \delta t_2 - \delta t_2$$

$$= t_2$$

(20)

$$dt' = dt - dt \frac{d}{dt}\delta t$$

$$= dt\left[1 - \frac{d}{dt}\delta t\right]$$

(21)

$$t = t' + \delta t(t)$$

$$= t' + \delta t(t' + \delta t)$$

$$= t' + \delta t(t') + O(\delta t)^2$$

$$dt = dt' + dt' \frac{d}{dt'}\delta t$$

$$= dt'\left[1 + \frac{d}{dt'}\delta t\right]$$

(22) Note that

$$\frac{d}{dt}\,\delta t(t) = \frac{d}{dt}\,\delta t\,(t'+\delta t)$$

$$= \frac{d}{dt}\,\delta t(t')$$

$$= \frac{dt'}{dt}\,\frac{d}{dt'}\,\delta t(t')$$

$$= \frac{d}{dt'}\,\delta t\,(t') \qquad + O(\delta t)^2.$$

(23) Further, for consistency, we check, using (21)

$$dt' = \frac{dt}{\left[1+\frac{d}{dt'}\,\delta t\right]}$$

$$= dt\left[1-\frac{d}{dt'}\,\delta t\right]$$

$$= dt\left[1-\frac{d}{dt}\,\delta t\right] \qquad \text{using (22)}$$

which is (20). Thus, all is fine.

(24)

$$\frac{d}{dt} = \frac{dt'}{dt}\,\frac{d}{dt'}$$

$$= \left[1-\frac{d}{dt}\,\delta t\right]\frac{d}{dt'}$$

$$= \left[1-\frac{d}{dt'}\,\delta t\right]\frac{d}{dt'}$$

25 Using 18 to 24 in 17

$$W + \delta W = \int_{t_1}^{t_2} dt' \left[1 + \frac{d}{dt'}\delta t\right] L\left(x+\delta x, \left[1 - \frac{d}{dt'}\delta t\right]\frac{dx}{dt'} + \frac{d}{dt'}\delta x, t' + \delta t\right)$$

$$= \int_{t_1}^{t_2} dt \left[1 + \frac{d}{dt}\delta t\right] L\left(x+\delta x, \frac{dx}{dt} - \frac{dx}{dt}\frac{d}{dt}\delta t + \frac{d}{dt}\delta x, t + \delta t\right)$$

$$= \int_{t_1}^{t_2} dt \left[1 + \frac{d}{dt}\delta t\right]\left[L + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\left\{-\frac{dx}{dt}\frac{d}{dt}\delta t + \frac{d}{dt}\delta x\right\} + \frac{\partial L}{\partial t}\delta t\right]$$

$$\delta W = \int_{t_1}^{t_2} dt \left[L\frac{d}{dt}\delta t + \frac{\partial L}{\partial x}\delta x - \frac{\partial L}{\partial \dot{x}}\frac{dx}{dt}\frac{d}{dt}\delta t + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}\delta x + \frac{\partial L}{\partial t}\delta t\right]$$

$$= \int_{t_1}^{t_2} dt \frac{d}{dt}\left[L\,\delta t - \frac{\partial L}{\partial \dot{x}}\frac{dx}{dt}\delta t + \frac{\partial L}{\partial \dot{x}}\delta x\right]$$

$$+ \int_{t_1}^{t_2} dt \left[-\frac{dL}{dt}\delta t + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\frac{dx}{dt}\right)\delta t - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\delta x \right.$$

$$\left. + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial t}\delta t\right]$$

$$= \int_{t_1}^{t_2} dt \left[\delta x \overbrace{\frac{\partial L}{\partial \dot{x}}}^{P} - \delta t \overbrace{\left\{\frac{\partial L}{\partial \dot{x}}\frac{dx}{dt} - L\right\}}^{H}\right]$$

$$+ \int_{t_1}^{t_2} dt\, \delta t \left[\frac{d}{dt}\overbrace{\left\{\frac{\partial L}{\partial \dot{x}}\frac{dx}{dt} - L\right\}}^{H} + \frac{\partial L}{\partial t}\right] \qquad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

$$+ \int_{t_1}^{t_2} dt\, \delta x \left[\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right]$$

(26) Principle of stationary action states that

$$\delta x = 0 \quad \text{at} \quad t_1 \quad \text{and} \quad t_2 ,$$

$$\delta t = 0 \quad \text{at} \quad t_1 \quad \text{and} \quad t_2 ,$$

then using the definition

$$p = \frac{\partial L}{\partial \dot{x}}$$

$$H = \frac{\partial L}{\partial \dot{x}} \frac{dx}{dt} - L .$$

we have the equations of motion

$$\frac{dp}{dt} = \frac{\partial L}{\partial x}$$

and

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t} .$$

(27) Thus, for the correct path

$$\delta W = \int_{t_1}^{t_2} dt \frac{d}{dt} G \qquad G = p\,\delta x - H\,\delta t$$

such that

$$\delta W_{12} = G_2 - G_1$$

(28) Rigid translation:

$$\delta x = \delta \epsilon$$

$$\delta t = 0$$

$$\delta W = \int_{t_1}^{t_2} dt \, \frac{d}{dt} (p \, \delta\epsilon)$$

$$= \int_{t_1}^{t_2} dt \, \delta\epsilon \, \frac{dp}{dt}$$

$\Rightarrow$ if $\delta W = 0$ then $\frac{dp}{dt} = 0$.

This is the connection between conservation of momentum and rigid translation.

(29) Rigid translation in time:

$$\delta x = 0$$

$$\delta t = \delta \epsilon^0$$

$\Rightarrow$ if $\delta W = 0$ then $\frac{dH}{dt} = 0$

Conservation of energy is generated by rigid translation in time.

(30) Rigid rotations $\longleftrightarrow$ conservation of angular momentum.

$\rightarrow$ HW.