SET-02

Canonical transformations

(1) The stationary action principle states that for any dynamical system there exists a Lagrangian, $\dot{q} = \frac{dq}{dt}$

$$L(q, \dot{q}, t),$$

which satisfies the Hamilton's principle

$$\delta \int_{t_1}^{t_2} dt \; L(q, \dot{q}, t) = 0 .$$

(2) Using the definitions

(i) $$p = \frac{\partial L}{\partial \dot{q}}$$

(ii) $$H = p\dot{q} - L = H(q, p, t)$$

the Hamilton's principle can be restated as

$$\delta \int_{t_1}^{t_2} dt \left[p\dot{q} - H \right] = 0 .$$

(3) Note that, assumed $H = H(q,p)$

$$0 = \delta \int_{t_1}^{t_2} dt \left[p\dot{q} - H \right] = \int_{t_1}^{t_2} dt \left[\delta p \, \dot{q} + p \, \delta\dot{q} - \frac{\partial H}{\partial q} \delta q - \frac{\partial H}{\partial p} \delta p \right]$$

$$= \int_{t_1}^{t_2} dt \left[- \delta q \left(\frac{dp}{dt} + \frac{\partial H}{\partial q} \right) + \delta p \left(\frac{dq}{dt} - \frac{\partial H}{\partial p} \right) \right]$$

implies the equations of motion

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = - \frac{\partial H}{\partial q} .$$

(4) Canonical transformations:

$$Q = Q(q, p, t)$$
$$\mathbb{P} = \mathbb{P}(q, p, t)$$

are transformations that preserve the Hamilton's principle,

$$\delta \int_{t_1}^{t_2} dt \left[\mathbb{P}\dot{Q} - K \right] = 0 ,$$

$H(q, p, t) \rightarrow K(\mathbb{P}, Q, t)$

which in turns implies the transformed equations of motion

$$\frac{dQ}{dt} = \frac{\partial H}{\partial \mathbb{P}}$$
$$\frac{d\mathbb{P}}{dt} = -\frac{\partial H}{\partial Q} .$$

(5) What are requirements for the Hamilton's principle to be preserved?

$$\delta \int_{t_1}^{t_2} dt \left[p\dot{q} - H(q,p,t) \right] = 0$$

$\rightarrow$ variation in q, p; $\delta q, \delta p$ are $= 0$ at t_1 and t_2.

$$\delta \int_{t_1}^{t_2} dt \left[\mathbb{P}\dot{Q} - K(Q, \mathbb{P}, t) \right] = 0$$

$\rightarrow$ variation in $Q, \mathbb{P}$; δQ and $\delta \mathbb{P} = 0$ at t_1 and t_2.

$$\delta \int_{t_1}^{t_2} dt \left[\mathbb{P}\dot{Q} \right.$$

We always have.

$$\delta \int_{t_1}^{t_2} dt \, \frac{d}{dt} W(q, p; t; Q, \mathbb{P}) = 0$$

because $\delta q, \delta p, \delta Q$, and $\delta \mathbb{P}$ are zero at t_1 and t_2.

6 Thus, the conditions for the transformations in 4 to be canonical is

$$p\,dq - H\,dt = \mathbb{P}\,dQ - K\,dt + dW$$

$$= \mathbb{P}\,dQ - K\,dt + \frac{\partial W}{\partial q}dq + \frac{\partial W}{\partial p}dp + \frac{\partial W}{\partial t}dt + \frac{\partial W}{\partial \mathbb{P}}d\mathbb{P} + \frac{\partial W}{\partial Q}dQ$$

which implies

(i) dq: $\dfrac{\partial W}{\partial q} = p$

(ii) dp: $\dfrac{\partial W}{\partial p} = 0$

(iii) dQ: $\dfrac{\partial W}{\partial Q} = -\mathbb{P}$

(iv) dQ: $\dfrac{\partial W}{\partial \mathbb{P}} = 0$

(v) dt: $\dfrac{\partial W}{\partial t} = K - H$

7 From 6-(ii) and 6-(iv) we conclude that W is independent of p and $\mathbb{P}$. Thus,

$$W = W(q, Q, t)$$

and the conditions for the transformations to be canonical are

(i) $\dfrac{\partial W}{\partial t} = K - H$

(ii) $\dfrac{\partial W}{\partial q} = p$

(iii) $\dfrac{\partial W}{\partial Q} = -\mathbb{P}$.

8 Thus, if the transformations

$$Q = Q(q, p, t)$$
$$\mathbb{P} = \mathbb{P}(q, p, t)$$

are canonical, then there exists a generating function,

$$W = W(q, Q, t),$$

such that

$$dW = \frac{\partial W}{\partial t} dt + \frac{\partial W}{\partial q} dq + \frac{\partial W}{\partial Q} dQ$$
$$= (K - H)\, dt + p\, dq - \mathbb{P}\, dQ$$

9 Hamilton - Jacobi equation

If we choose the transformation such that

$$K = 0,$$

we will have (using the transformed equations of motion)

$$\frac{dQ}{dt} = 0$$
$$\frac{d\mathbb{P}}{dt} = 0.$$

Thus, the new dynamical variables, Q and $\mathbb{P}$, are constants of motion.

(10) Using (9) in (8), in this particular case we have.

$$dW = \underbrace{(K-H)}_{=0} dt + p\,dq - \underbrace{\mathbb{P}\,dQ}_{=0}$$

Thus, the generating function is a func^n of q and t only,

$$W = W(q,t)$$

such that

$$dW = \frac{\partial W}{\partial t} dt + \frac{\partial W}{\partial q} dq = -H\,dt + p\,dq.$$

More explicitly,

$$W = W(q,t\,;\underbrace{Q}_{\text{constant of motion}})$$

The conditions

(i) $\dfrac{\partial W}{\partial t} = -H$

(ii) $\dfrac{\partial W}{\partial q} = p$

are called Hamilton-Jacobi equation.

→ Note that $\dfrac{\partial W}{\partial Q} = -\mathbb{P} = \text{constant of motion}.$

(11) Let us consider the example of harmonic oscillator described by the Hamiltonian

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}kx^2 \qquad \frac{k}{m} = \omega^2$$

$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

Define dimensionless variables

$$p \to p\sqrt{m} \quad \text{and} \quad x \to \frac{x}{\omega\sqrt{m}}.$$

Then

$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}x^2$$

(12) Equations of motion are.

(i) $\frac{dx}{dt} = \frac{\partial H}{\partial p} = p$

(ii) $\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -kx.$

Together they imply

dimensionless time $t \to \omega t$

$$\frac{d^2x}{dt^2} = -x \quad \Rightarrow \quad x(t) = c_1 \sin(t + c_2)$$

$$p(t) = \frac{dx}{dt}$$

$$= c_1 \cos(t + c_2)$$

$$H(x,p) = \frac{1}{2}c_1^2 = \text{Energy} = E$$

$$\Rightarrow c_1 = \sqrt{2E}$$

Thus,

$$x(t) = \sqrt{2E}\,\sin(t + c_2)$$

13 Let us consider

$$p = f(\mathbb{P}) \cos Q \qquad p^2 + x^2 = f(\mathbb{P})^2$$

$$x = f(\mathbb{P}) \sin Q \qquad \frac{x}{p} = \tan Q$$

Our goal will be to find the particular $f(\mathbb{P})$ that will let the above transformation be canonical.

14 $$K(\mathbb{P}, Q) = \frac{1}{2}[f(\mathbb{P})]^2$$

15 Requirement for canonical transformation is the existance of the generating function $W(x, Q, t)$ that satisfies

(i) $\frac{\partial W}{\partial t} = K - H \rightarrow \frac{\partial W}{\partial t} = 0$

(ii) $\frac{\partial W}{\partial x} = p \rightarrow \frac{\partial W}{\partial x} = \frac{x}{\tan Q}$

(iii) $\frac{\partial W}{\partial Q} = -\mathbb{P} \rightarrow \frac{\partial W}{\partial Q} = -f^{-1}\left(\frac{x}{\sin Q}\right)$ using $x = f(\mathbb{P}) \sin Q$.

16 15-(i) immediately implies that W is time independent.
Using 15-(ii) we further have, integrating with x,

$$W(x, Q) = \frac{1}{2}\frac{x^2}{\tan Q} + C(Q)$$

$C(Q)$: integration constant.

17 Using 16 in 15 - (iii)

$$\frac{\partial}{\partial Q}\left[\frac{1}{2}\frac{x^2}{\tan Q} + c(Q)\right] = -f^{-1}\left(\frac{x}{\sin Q}\right)$$

$$-\frac{1}{2}\frac{x^2}{\sin^2 Q} + \frac{\partial c}{\partial Q} = -\mathbb{P} = -f^{-1}\left(\frac{x}{\sin Q}\right)$$

$$-\frac{1}{2}\left[f(\mathbb{P})\right]^2 + \frac{\partial c(Q)}{\partial Q} = -\mathbb{P}$$

$$\Rightarrow \quad f(\mathbb{P}) = \sqrt{2\mathbb{P} + c_3}$$

requiring $c_3 = \frac{\partial c(Q)}{\partial Q}$ = constant.

18 Thus, we have

$$K(\mathbb{P}, Q) = \frac{1}{2}\left[f(\mathbb{P})\right]^2 = \mathbb{P} + \frac{c_3}{2} \equiv E = \text{Energy}$$

Equations of motion

(i) $\frac{dQ}{dt} = \frac{\partial K}{\partial \mathbb{P}} = 1 \quad \Rightarrow \quad Q(t) = t + c_2$

(ii) $\frac{d\mathbb{P}}{dt} = -\frac{\partial K}{\partial Q} = 0 \quad \Rightarrow \quad \mathbb{P} = \text{constant} = \sqrt{2E - c_3}$

19 Using 13 in 18 -(i) we have

$$Q(t) = t + c_2 \quad \Rightarrow \quad \sin^{-1}\left(\frac{x}{f(\mathbb{P})}\right) = t + c_2$$

$$\Rightarrow \quad x(t) = f(\mathbb{P}) \sin(t + c_2) = \sqrt{2\mathbb{P} + c_3}\, \sin(t + c_2) = \sqrt{2E}\, \sin(t + c_2)$$

which is consistent with 12. ✓

(20) Poisson bracket :

For two function

$$F = F(q, p, t)$$

$$G = G(q, p, t)$$

the Poisson bracket is defined as

$$[F, G] = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

(21) Few immediate evaluations are.

(i) $[q, q] = 0$ $\qquad \frac{\partial q}{\partial q}\underbrace{\frac{\partial q}{\partial p}}_{=0} - \underbrace{\frac{\partial q}{\partial p}}_{=0}\frac{\partial q}{\partial q} = 0$

(ii) $[p, p] = 0$ $\qquad \underbrace{\frac{\partial p}{\partial q}}_{=0}\frac{\partial p}{\partial p} - \frac{\partial p}{\partial p}\underbrace{\frac{\partial p}{\partial q}}_{=0} = 0$

(iii) $[q, p] = 1$ $\qquad \underbrace{\frac{\partial q}{\partial q}}_{=1}\underbrace{\frac{\partial p}{\partial p}}_{=1} - \underbrace{\frac{\partial q}{\partial p}}_{=0}\underbrace{\frac{\partial p}{\partial q}}_{=0} = 1$

(22) Let us now show that the Poisson brack is invariant under canonical transformation, that is

$$[F, G]_{p,q} = [F, G]_{P,Q}$$

(23) Let

we shall restrict the proof to time independent situations here.

$$Q = Q(q, p)$$

$$\mathbb{P} = \mathbb{P}(q, p)$$

a canonical transformation. Thus, there exists

$$W = W(q, Q, t)$$

such that

$$dW = (K - H)\, dt + p\, dq - \mathbb{P}\, dQ + 0\, dp + 0\, d\mathbb{P}$$

(24) Using

$$dQ = \frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp$$

in (23) we have

$$dW = (K - H)\, dt + p\, dq - \mathbb{P}\left[\frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial p} dp\right]$$

$$= (K - H)\, dt + \left[p - \mathbb{P}\frac{\partial Q}{\partial q}\right] dq - \mathbb{P}\frac{\partial Q}{\partial p} dp$$

which states that

(i) $$\frac{\partial W}{\partial t} = K - H$$

(ii) $$\frac{\partial W}{\partial q} = p - \mathbb{P}\frac{\partial Q}{\partial q}$$

(iii) $$\frac{\partial W}{\partial p} = -\mathbb{P}\frac{\partial Q}{\partial p}$$

(25) The integrability condition is

$$\frac{\partial}{\partial q}\frac{\partial}{\partial p} W = \frac{\partial}{\partial p}\frac{\partial}{\partial q} W$$

which implies

$$\frac{\partial}{\partial q}\left[-\mathbb{P}\frac{\partial Q}{\partial p}\right] = \frac{\partial}{\partial p}\left[p - \mathbb{P}\frac{\partial Q}{\partial q}\right]$$

$$-\frac{\partial \mathbb{P}}{\partial q}\frac{\partial Q}{\partial p} - \mathbb{P}\frac{\partial^2 Q}{\partial q\,\partial p} = 1 - \frac{\partial \mathbb{P}}{\partial p}\frac{\partial Q}{\partial q} - \mathbb{P}\frac{\partial^2 Q}{\partial p\,\partial q}$$

✓ ✓

$$\Rightarrow \quad [Q, \mathbb{P}]_{q,p} = 1$$

(26) Now,

$$[F, G]_{q,p} = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

$$= \left(\frac{\partial F}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial q}\right)\left(\frac{\partial G}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial G}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial p}\right) - \left(\frac{\partial F}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial p}\right)\left(\frac{\partial G}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial G}{\partial \mathbb{P}}\frac{\partial \mathbb{P}}{\partial q}\right)$$

$$= \frac{\partial F}{\partial Q}\frac{\partial G}{\partial Q}\underbrace{[Q, Q]_{q,p}}_{=0} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial \mathbb{P}}\underbrace{[\mathbb{P}, \mathbb{P}]_{q,p}}_{=0} + \frac{\partial F}{\partial Q}\frac{\partial G}{\partial \mathbb{P}}[Q, \mathbb{P}]_{q,p} + \frac{\partial F}{\partial \mathbb{P}}\frac{\partial G}{\partial Q}[\mathbb{P}, Q]_{q,p}$$

$$= [F, G]_{Q,\mathbb{P}}\underbrace{[Q, \mathbb{P}]_{q,p}}_{=1}$$

$$= [F, G]_{Q,\mathbb{P}}$$

27 If one of the functions is the Hamiltonian

$$[F, H] = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q}$$

$$= \frac{\partial F}{\partial q}\frac{dq}{dt} + \frac{\partial F}{\partial p}\frac{dp}{dt}$$

$$= \frac{dF}{dt} - \frac{\partial F}{\partial t}$$

28 Thus we have

$$\frac{dq}{dt} = [q, H]$$

$$\frac{dp}{dt} = [p, H]$$

$\because \frac{\partial q}{\partial t} = 0$ and $\frac{\partial p}{\partial t} = 0$

→ have no explicit time dependence.

29 Thus, the equations of motion are

$$\frac{dq}{dt} = [q, H] = \frac{\partial H}{\partial p}$$

$$\frac{dp}{dt} = [p, H] = -\frac{\partial H}{\partial q}$$

If $F(q,p)$ and H are algebraic functions of q and p, all Poisson brackets can be reduced to fundamental Poisson brackets, $[q,q]$, $[p,p]$, and $[q,p]$, by algebraic processes.

Poisson brackets can be solved without solving the differential equation.

Examples: (i) Harmonic oscillator can be solved in this manner.

(ii) Pauli solved the hydrogen atom in this manner.

(iii) Dirac's book and Schwinger's book use it a lot.

30 (Lie) algebra satisfied by Poisson bracket

(i) $[F, G] = -[G, F]$

(ii) $[aF_1 + bF_2, G] = a[F_1, G] + b[F_2, G]$

(iii) $[F_1 F_2, G] = F_1 [F_2, G] + [F_1, G] F_2$

(iv) $[[F, G], H] + [[G, H], F] + [[H, F], G] = 0$ → Jacobi identity.

31 Multiplication of numbers is associative,

$$(ab)c - a(bc) = 0.$$

For Poisson brackets we have

$$[[F, G], H] - [F, [G, H]] = [[F, G], H] + [[G, H], F]$$
$$= -[[H, F], G]$$

Thus, Jacobi identity is a statement of non-associativity of an operation.

HW → 32 In homework you will show that the operation of matrix multiplication,

$$[A, B]_{\text{Matrix}} = AB - BA$$

satisfies Lie algebra.

(33) Jacobi identity

$$[F, G] = \frac{\partial F}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial q}$$

$$\frac{\partial F}{\partial q} \equiv F_1 \qquad \frac{\partial F}{\partial p} \equiv F_2$$

$$= F_1 G_2 - F_2 G_1$$

$$[[F,G], H] = (F_1 G_2 - F_2 G_1)_1 H_2 - (F_1 G_2 - F_2 G_1)_2 H_1$$

$$= (F_{11} G_2 + F_1 G_{21} - F_{21} G_1 - F_2 G_{11}) H_2$$

$$- (F_{12} G_2 + F_1 G_{22} - F_{22} G_1 - F_2 G_{12}) H_1$$

$$= F_{11} G_2 H_2 + F_1 G_{21} H_2 - F_{21} G_1 H_2 - F_2 G_{11} H_2$$

$$- F_{12} G_2 H_1 - F_1 G_{22} H_1 + F_{22} G_1 H_1 + F_2 G_{12} H_1$$

$$= - F_1 G_{22} H_1 + F_1 G_{21} H_2 + F_2 G_{12} H_1 - F_2 G_{11} H_2$$

$$+ G_1 F_{22} H_1 - G_1 F_{21} H_2 - G_2 F_{12} H_1 + G_2 F_{11} H_2$$

$$= (F_1 \;\; F_2) \begin{pmatrix} -G_{22} & G_{21} \\ G_{12} & -G_{11} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} + (G_1 \;\; G_2) \begin{pmatrix} F_{22} & -F_{21} \\ -F_{12} & F_{11} \end{pmatrix} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$$

define

$$= - \vec{F} \cdot \overleftrightarrow{G} \cdot \vec{H} + \vec{G} \cdot \overleftrightarrow{F} \cdot \vec{H}$$

$$\left.\begin{array}{l}\text{LHS of} \\ \text{Jacobi identity}\end{array}\right\} = - \vec{F} \cdot \overleftrightarrow{G} \cdot \vec{H} + \vec{G} \cdot \overleftrightarrow{F} \cdot \vec{H}$$

$$- \vec{G} \cdot \overleftrightarrow{H} \cdot \vec{F} + \vec{F} \cdot \overleftrightarrow{H} \cdot \vec{G}$$

$$- \vec{H} \cdot \overleftrightarrow{F} \cdot \vec{G} + \vec{H} \cdot \overleftrightarrow{G} \cdot \vec{F}$$

$$= 0 \quad \text{using property of trace.}$$

34 Connection between CM and QM:

$$[F, G]_{\text{Poisson}} \longleftrightarrow \frac{1}{i\hbar} [F, G]_{\text{commutation}}$$

F, G are classical functions

F, G are quantum operators.

→ called the correspondence principle.

→ one to one relation implied by Lie algebra.