SET - 04

## Linear vector space over complex field

1. Vector space.

$$\psi \equiv (x_1, x_2, \ldots, x_n)$$

— $x_i$ are ordered
— $x_i$ are complex

(i) $\psi + \phi = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$

(ii) $(\psi_1 + \psi_2) + \psi_3 = \psi_1 + (\psi_2 + \psi_3)$

(iii) $\psi + 0 = \psi$

(iv) $\psi + (-\psi) = 0$

vector addition

(i) $\lambda \psi = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

(ii) $\lambda \cdot 1 \cdot \psi = \lambda \psi$ and $\lambda \frac{1}{\lambda} \psi = 1 \psi$

(iii) $\lambda (\psi + \phi) = \lambda \psi + \lambda \phi$

(iv) $(\lambda_1 + \lambda_2) \psi = \lambda_1 \psi + \lambda_2 \psi$

scalar multiplication

2. One can introduce base vectors

$$e_1 = (1, 0, \ldots, 0)$$
$$e_2 = (0, 1, \ldots, 0)$$
$$\vdots$$
$$e_n = (0, 0, \ldots, 1)$$

→ $(e_i, e_j) = \delta_{ij}$
orthonormal basis set

→ Summation convention

$$\psi = e_1 x_1 + e_2 x_2 + \ldots + e_n x_n$$
$$= \sum_{i=1}^{n} e_i x_i \equiv e_i x_i$$

3 **Scalar product**

One could define a scalar product on a vector space.

Properties of scalar product:

(i) $(\phi, \psi) = (\psi, \phi)^*$

(ii) $(\phi, \lambda\psi) = \lambda(\phi, \psi) \Rightarrow (\lambda\phi, \psi) = \lambda^*(\phi, \psi)$

(iii) $(\phi, \psi_1 + \psi_2) = (\phi, \psi_1) + (\phi, \psi_2)$

(iv) $(e_i, e_j) = \delta_{ij}$

4 Notes:

→ Using 3 (i) we have $(\psi, \psi) = (\psi, \psi)^* =$ real.

$\phi = e_i x_i$

$\psi = e_j y_j$

→ $$(\phi, \psi) = (e_i x_i, e_j y_j)$$
$$= x_i^* y_j (e_i, e_j)$$
$$= x_i^* y_j \delta_{ij}$$
$$= x_i^* y_i$$

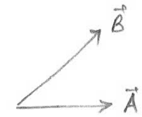
→ This is a generalization of the scalar product of two vectors in Euclidean vector space.

$$\vec{A} \cdot \vec{B} = a_i b_i$$

→ $(\psi, \psi) = x_i^* x_i = \sum_i |x_i|^2 \geq 0$

→ $(\psi, \psi) = 0 \Rightarrow \psi = 0$

## 5 Schwarz's inequality:




$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos\theta$$

$$\frac{(\vec{A}\cdot\vec{B})^2}{(\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B})} = \cos^2\theta \leq 1$$

Let $\alpha = \lambda\psi + \mu\phi$

$$(\alpha,\alpha) = (\lambda\psi + \mu\phi,\ \lambda\psi + \mu\phi)$$

$$= |\lambda|^2(\psi,\psi) + \lambda^*\mu(\psi,\phi) + \lambda\mu^*(\phi,\psi) + |\mu|^2(\phi,\phi) \qquad \text{(iii)}$$

For what $\lambda$ is $(\alpha,\alpha)$ minimised?

$$\frac{\partial}{\partial\lambda}(\alpha,\alpha) = \lambda^*(\psi,\psi) + \mu^*(\phi,\psi)$$

$$\frac{\partial}{\partial\lambda^*}(\alpha,\alpha) = \lambda(\psi,\psi) + \mu(\psi,\phi)$$

$$= \left[\lambda^*(\psi,\psi) + \mu^*(\phi,\psi)\right]^* = \left[\frac{\partial}{\partial\lambda}(\alpha,\alpha)\right]^*$$

$$\frac{\partial}{\partial\mu}(\alpha,\alpha) = \lambda^*(\psi,\phi) + \mu^*(\phi,\phi) = \left[\frac{\partial}{\partial\mu^*}(\alpha,\alpha)\right]^*$$

Setting $\frac{\partial}{\partial\lambda^*}(\alpha,\alpha) = 0 \Rightarrow \lambda(\psi,\psi) = -\mu(\psi,\phi)$

$$\Rightarrow \lambda = -\mu\frac{(\psi,\phi)}{(\psi,\psi)} \qquad \text{(i)}$$

$$\Rightarrow \lambda^* = -\mu^*\frac{(\phi,\psi)}{(\psi,\psi)} \qquad \text{(ii)}$$

$$\Rightarrow$$

Using (i) and (ii) in (iii)

$$(\alpha,\alpha) = \mu\mu^*\frac{(\psi,\phi)}{(\psi,\psi)}\frac{(\phi,\psi)}{(\psi,\psi)}(\psi,\psi) - \mu\mu^*\frac{|(\psi,\phi)|^2}{(\psi,\psi)} - \mu\mu^*\frac{|(\psi,\phi)|^2}{(\psi,\psi)} + \mu\mu^*(\phi,\phi)$$

$$= \mu\mu^*\left[(\phi,\phi) - \frac{|(\psi,\phi)|^2}{(\psi,\psi)}\right]$$

Since $(\alpha,\alpha) \geqslant 0$ we have

$$\frac{|(\psi,\phi)|^2}{(\psi,\psi)(\phi,\phi)} \leqslant 1.$$

6 We can define an angle.

$$\cos^2\theta = \frac{|(\psi, \phi)|^2}{(\psi,\psi)(\phi,\phi)} \leq 1$$

$\cos\theta\big|_{\psi,\phi} = 1 \Rightarrow \psi$ and $\phi$ are in the same direction.

$\psi = a\phi$

$\cos\theta\big|_{\psi,\phi} = 0 \Rightarrow \psi$ and $\phi$ are orthogonal.

7 <u>Change of base vectors</u>

$\psi = e_i x_i$

$\psi = e_i' x_i'$

$(e_i, e_j) = \delta_{ij}$

$(e_i', e_j') = \delta_{ij}$ → orthonormal basis

Let $e_i = S_{ij} e_j'$

Clearly $(e_j', e_i) = (e_j', S_{ik} e_k') = S_{ij}$

$\Rightarrow S_{ij}$ are direction cosines between $e_i$ and $e_j'$

$(e_i, e_j) = (S_{im} e_m', S_{jn} e_n') = S_{im}^* S_{jm}$ $\qquad S^\dagger = (S^T)^*$

$\Rightarrow S S^\dagger = 1$

Similarly, $e_i' = S_{ij}^* e_j$

because $(e_i', e_j) = (S_{ik}^* e_k, e_j) = S_{ij}$

8 Thus, we have.

$$\vec{e} = \overleftrightarrow{S}\,\vec{e}\,'$$
$$\vec{e}\,' = \overleftrightarrow{S}^{\dagger}\,\vec{e}$$

9 Since $SS^{\dagger} = 1$

we have

$$(\det S)(\det S^{\dagger}) = 1 \qquad \because \det A = \det A^{T}$$
$$(\det S)(\det S^{*}) = 1$$
$$(\det S)(\det S)^{*} = 1$$
$$|(\det S)|^{2} = 1$$

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$$(\psi, \phi) = (e_i x_i \,,\, e_j y_j) = x_i^{*} y_i$$
$$= x_i^{*} y_j \,(e_i, e_j)$$
$$= x_i^{*} y_j \,(S_{im} e_m' \,,\, S_{jn} e_n')$$
$$= x_i^{*} y_j \, S_{im}^{*} S_{jn} \,\delta_{mn}$$
$$= x_i^{*} y_j \, S_{jm} S_{mi}^{\dagger} \quad ✓$$
$$= x_i^{*} y_i$$