SET-05

Linear transformation on vector space

1. Consider a mapping from vector space to itself.

$$\psi \longrightarrow \psi' = A\psi$$

2. Properties:

(i) $A(\lambda\psi) = \lambda(A\psi)$

(ii) $A(\psi_1 + \psi_2) = A\psi_1 + A\psi_2$

3. Linear transformations can be characterized by their action on the base vectors

$$A e_i = e_k A_{ki}$$

Thus,

$$A\psi = A e_i x_i$$
$$= e_j A_{ji} x_i$$

$A e_i = e_k A_{ki}$
(transformat of basis)

Note that,

$$e_i x_i' = \psi' = A\psi$$
$$= A e_j x_j$$
$$= e_i A_{ij} x_j$$

$$\Rightarrow \quad x_i' = A_{ij} x_j$$

$x_i' = A_{ij} x_j$
(transformation of vector)

4 $$(e_i, A e_j) = (e_i, e_k A_{kj}) = A_{ij}$$

5 $$(\phi, A\psi) = (e_i x_i, A e_j y_j) = x_i^* A_{ij} y_j$$

6 Addition operation

$$C = A + B$$

$$C\psi = (A+B)\psi = A\psi + B\psi$$

$$C e_j = (A+B) e_j = A e_j + B e_j$$

$$e_k A_{kj} = e_k (A+B)_{kj} = e_k A_{kj} + e_k B_{kj}$$

$$(A+B)_{kj} = A_{kj} + B_{kj}$$

7 Product operation

$$(AB)\psi = A(B\psi)$$

Choosing $C = AB$ and $\psi = e_j$

$$C e_j = e_k C_{kj} = e_k (AB)_{kj}$$

$$C e_j = (AB) e_j = A(B e_j) = A e_i B_{ij} = e_k A_{ki} B_{ij}$$

$$\Rightarrow (AB)_{kj} = A_{ki} B_{ij}$$

8 Adjoint operator

$$(\phi, A\psi) = (A^\dagger \phi, \psi)$$

$A^\dagger$ is the adjoint operator corresponding to A.

Using $\phi = e_i$ and $\psi = e_j$ we have

$$(e_i, Ae_j) = (A^\dagger e_i, e_j)$$
$$= (e_j, A^\dagger e_i)^*$$

$$A_{ij} = (A^\dagger_{ji})^*$$

$$\Rightarrow \quad A^\dagger_{ij} = A^*_{ji}$$

9 $$(\phi, AB\psi) = ((AB)^\dagger \phi, \psi) = (B^\dagger A^\dagger \phi, \psi)$$

$$\Rightarrow \quad (AB)^\dagger = B^\dagger A^\dagger$$

10 Hermitian operator (self adjoint operator)

$$A = A^\dagger$$

11 One can always construct Hermitian operator from non-Hermitian operators.

$$A = K + K^\dagger \qquad (K \text{ is not necessarily Hermitian})$$

$$A^\dagger = K^\dagger + K = A$$

12 Another way to construct a Hermitian operator is

$$A = \frac{1}{i}(K - K^\dagger)$$

$$A^\dagger = -\frac{1}{i}(K^\dagger - K) = A$$

This mean

$$(\lambda A)^\dagger = \lambda^* A^\dagger$$

13 If $A^\dagger = A$

then $(\phi, A\psi) = (A\phi, \psi)$ for any ϕ and ψ.

14 Clearly,

$$(\psi, A\psi) = (A\psi, \psi) = (\psi, A\psi)^*$$

if A is Hermitian.

$\Rightarrow$ $(\psi, A\psi)$ is real

15 <u>Unitary operator</u>

$$(U\phi, U\psi) = (\phi, \psi)$$

$$\Rightarrow \quad UU^\dagger = 1 \quad \Rightarrow \quad U^{-1} = U^\dagger$$

16 We can show that the requirement

$$(U\psi, U\psi) = (\psi, \psi)$$

implies

$$(U\phi, U\psi) = (\phi, \psi).$$

<u>Proof</u>:

Let $\alpha = \lambda\psi + \mu\phi$

Then $(U\alpha, U\alpha) = (\alpha, \alpha)$

$$\left(U(\lambda\psi+\mu\phi), U(\lambda\psi+\mu\phi)\right) = (\lambda\psi+\mu\phi, \lambda\psi+\mu\phi)$$

$$\lambda\lambda^*(U\psi, U\psi) + \lambda^*\mu(U\psi, U\phi) + \mu^*\lambda(U\phi, U\psi) + \mu\mu^*(U\phi, U\phi)$$

$$= \lambda\lambda^*(\psi,\psi) + \lambda^*\mu(\psi,\phi) + \mu^*\lambda(U\phi, U\psi) + \mu\mu^*(\phi,\phi)$$

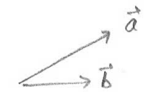
$$\lambda^*\mu(U\psi, U\phi) + \mu^*\lambda(U\phi, U\psi) = \lambda^*\mu(\psi,\phi) + \mu^*\lambda(\phi,\psi)$$

$$\lambda^*\mu\left[(U\psi, U\phi) - (\psi,\phi)\right] + \mu^*\lambda\left[(U\phi, U\psi) - (\phi,\psi)\right] = 0$$

Since this is true for arbitrary λ and μ we have

$$(U\phi, U\psi) = (\phi, \psi).$$

(17) <u>Projection operator</u>



$$P_\phi \, \psi = \phi \, (\phi, \psi)$$

$$P_{\vec{a}} \, \vec{b} = \vec{a} \, (\vec{a} \cdot \vec{b})$$

(18) P_α is Hermitian.

$$(\phi, P_\alpha \psi) = (\phi, \alpha(\alpha, \psi)) = (\phi, \alpha)(\alpha, \psi)$$

$$(P_\alpha \phi, \psi) = (\alpha(\alpha, \phi), \psi) = (\alpha, \phi)^* (\alpha, \psi) = (\phi, \alpha)(\alpha, \psi).$$

$$\Rightarrow \quad (\phi, P_\alpha \psi) = (P_\alpha \phi, \psi)$$

$$\Rightarrow \quad P_\alpha = P_\alpha^\dagger$$

(19) Projection operator associated with basis vector.

$$P_{e_i} \equiv P_i$$

$$P_i e_j = e_{\underline{i}} (e_i, e_j) = e_{\underline{i}} \delta_{ij}$$

(20)
$$P_i \psi = P_i e_j x_j = e_{\underline{i}} (e_i, e_j) x_j = e_{\underline{i}} \delta_{ij} x_j = e_{\underline{i}} x_i$$

$\underline{i}$ – no summation on i

(21)
$$(e_m, P_i e_n) = (e_m, e_{\underline{i}} (e_i, e_n)) = (e_m, e_{\underline{i}})(e_i, e_n) = \delta_{m\underline{i}} \delta_{in}.$$

(22) $$P_i P_j = \delta_{ij} P_i = \begin{cases} 0 & i \neq j \\ P_i & i = j \end{cases}$$

$$\begin{aligned} P_i P_j \psi &= P_i P_j e_k x_k \\ &= P_i e_{\underline{j}} (e_j, e_k) x_k \\ &= e_{\underline{i}} (e_i, e_{\underline{j}}) (e_j, e_k) x_k \\ &= e_{\underline{i}} \delta_{i\underline{j}} \delta_{jk} x_k \\ &= e_{\underline{i}} \delta_{i\underline{j}} x_j \\ &= \delta_{i\underline{j}} e_{\underline{j}} x_j && (\text{using } i=j, \text{ as per } \delta_{i\underline{j}}) \\ &= \delta_{i\underline{j}} P_j \psi. && (\text{using } (20)) \end{aligned}$$

(23) $$\sum_{i=1}^{n} P_i = 1$$

This is the statement of completeness relation.

(24) $$P_\alpha P_\beta P_\alpha = |(\alpha,\beta)|^2 P_\alpha = \begin{cases} 0 & \text{if } (\alpha,\beta) = 0 \\ P_\alpha & \text{if } (\alpha,\beta) = 1 \end{cases}$$

$$\begin{aligned} P_\alpha P_\beta P_\alpha \psi &= P_\alpha P_\beta \alpha (\alpha, \psi) \\ &= P_\alpha \beta (\beta, \alpha) (\alpha, \psi) \\ &= \alpha (\alpha, \beta) (\beta, \alpha) (\alpha, \psi) \\ &= |(\alpha,\beta)|^2 P_\alpha \psi. \end{aligned}$$