SET-08

Wave functions, Eigen basis, transformation function, and unitary operator in Stern-Gerlach experiment.

1

$$\vec{B} = B\hat{n}$$

$$\vec{\mu} = \frac{q}{2m}\vec{J} = \frac{q}{2m}\frac{\hbar}{2}\vec{\sigma}$$

2 Interaction energy $= H_{int} = -\vec{\mu}\cdot\vec{B} = -\frac{qB}{2m}\frac{\hbar}{2}\vec{\sigma}\cdot\hat{n}$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$H_{int} = -\frac{qB}{2m}\frac{\hbar}{2} H(\theta,\phi)$$

$$H(\theta,\phi) = \vec{\sigma}\cdot\hat{n} = \sigma_z\cos\theta + \sigma_x\sin\theta\cos\phi + \sigma_y\sin\theta\sin\phi$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

3 Pauli matrices

(i) $\sigma_i \sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\sigma_k$

$[A,B] = AB - BA$

$\{A,B\} = AB + BA$

(ii) $[\sigma_i, \sigma_j] = 2i\,\epsilon_{ijk}\sigma_k$

(iii) $\{\sigma_i, \sigma_j\} = 2\delta_{ij}$

(iv) $\sigma_i^2 = 1$, $Tr(\sigma_i) = 0$, $\det(\sigma_i) = -1$.

(v) $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b}) + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \Rightarrow (\vec{\sigma}\cdot\hat{n})^2 = 1.$

4 For a matrix M that satisfies $M^2 = 1$, we have

$$e^{i\alpha M} = \cos\alpha + iM\sin\alpha$$

Examples: (i) $e^{i\alpha\sigma_x} = \cos\alpha + i\sigma_x\sin\alpha.$

(ii) $e^{i\alpha\sigma_y} = \cos\alpha + i\sigma_y\sin\alpha$

(iii) $e^{i\alpha(\vec{\sigma}\cdot\hat{n})} = \cos\alpha + i(\vec{\sigma}\cdot\hat{n})\sin\alpha$

(5) $$H(\theta,\phi) = \vec{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$\lambda = \pm 1 \qquad \psi_+(\theta,\phi) = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \qquad \psi_-(\theta,\phi) = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix}$$

(6) $$H(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \qquad \psi_+(0,0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_-(0,0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H(\tfrac{\pi}{2},0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \qquad \psi_+(\tfrac{\pi}{2},0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \psi_-(\tfrac{\pi}{2},0) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$H(\tfrac{\pi}{2},\tfrac{\pi}{2}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \qquad \psi_+(\tfrac{\pi}{2},\tfrac{\pi}{2}) = \frac{1}{2}\begin{pmatrix} 1-i \\ 1+i \end{pmatrix} \qquad \psi_-(\tfrac{\pi}{2},\tfrac{\pi}{2}) = \frac{1}{2}\begin{pmatrix} -1+i \\ 1+i \end{pmatrix}$$

(7) Illustrative examples for probability amplitudes:

$$\left([+;0,0] \to [+;\tfrac{\pi}{2},0]\right) = \langle \psi_+(0,0) \,|\, \psi_+(\tfrac{\pi}{2},0)\rangle = (1\ 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

$[+;0,0]$: select $\sigma_z' = +1$; $[+;\frac{\pi}{2},0]$: select $\sigma_x' = +1$

$$\left([+;0,0] \to [-;\tfrac{\pi}{2},0]\right) = \langle \psi_+(0,0) \,|\, \psi_-(\tfrac{\pi}{2},0)\rangle = (1\ 0)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\left([-;0,0] \to [+;\tfrac{\pi}{2},0]\right) = \langle \psi_-(0,0) \,|\, \psi_+(\tfrac{\pi}{2},0)\rangle = (0\ 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\left([-;0,0] \to [-;\tfrac{\pi}{2},0]\right) = \langle \psi_-(0,0) \,|\, \psi_-(\tfrac{\pi}{2},0)\rangle = (0\ 1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(8) **Notation:**

Operators	eigen values
A	A'
a	a'

(9) **Wave function**

$$\psi_{n'}(q') = \langle q' | n' \rangle$$

Realization of eigenvector $|n'\rangle$ in the basis of $\langle q'|$.

Examples:

$$\psi_{\sigma_x'=+1}(\sigma_z') = \langle \sigma_z' | \sigma_x' = +1 \rangle$$

$$= \begin{pmatrix} \langle \psi_+(0,0) | \psi_+(\frac{\pi}{2},0) \rangle \\ \langle \psi_-(0,0) | \psi_+(\frac{\pi}{2},0) \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\psi_{\sigma_x'=-1}(\sigma_z') = \langle \sigma_z' | \sigma_x' = -1 \rangle$$

$$= \begin{pmatrix} \langle \psi_+(0,0) | \psi_-(\frac{\pi}{2},0) \rangle \\ \langle \psi_-(0,0) | \psi_-(\frac{\pi}{2},0) \rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(10) Transformation function : Expression of one basis in terms of another — Direction cosines.

Observe that :

$$| \psi_+ (\tfrac{\pi}{2},0)\rangle \langle \psi_+ (\tfrac{\pi}{2},0)| = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \;\; 1) \frac{1}{\sqrt{2}} = \frac{1}{2} (1 + \sigma_x)$$

$$| \psi_- (\tfrac{\pi}{2},0)\rangle \langle \psi_- (\tfrac{\pi}{2},0)| = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} (-1 \;\; 1) \frac{1}{\sqrt{2}} = \frac{1}{2} (1 - \sigma_x)$$

Further, note that

$$| \psi_+ (\tfrac{\pi}{2},0)\rangle \langle \psi_+ (\tfrac{\pi}{2},0)| + | \psi_- (\tfrac{\pi}{2},0)\rangle \langle \psi_- (\tfrac{\pi}{2},0)| = 1$$

(11) Thus, we can write

$$|\sigma_z'\rangle = 1 \, |\sigma_z'\rangle$$
$$= |\sigma_x'=+1\rangle \langle \sigma_x'=+1| \sigma_z'\rangle + |\sigma_x'=-1\rangle\langle\sigma_x'=-1|\sigma_z'\rangle$$

where we used

$$|\sigma_x'=+1\rangle\langle\sigma_x'=+1| + |\sigma_x'=-1\rangle\langle\sigma_x'=-1| = 1$$

12 In general we have.

$$|\psi_+(\theta,\phi)\rangle\langle\psi_+(\theta,\phi)| = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} & \sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \end{pmatrix}$$

$$|\sigma'_{\theta,\phi} = +1\rangle\langle\sigma'_{\theta,\phi} = +1| = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-i\phi} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{i\phi} & \sin^2\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+\cos\theta}{2} & \frac{1}{2}\sin\theta\, e^{-i\phi} \\ \frac{1}{2}\sin\theta\, e^{i\phi} & \frac{1-\cos\theta}{2} \end{pmatrix}$$

$$= \frac{1}{2}\left[1 + \vec{\sigma}\cdot\hat{n}\right]$$

$$|\psi_-(\theta,\phi)\rangle\langle\psi_-(\theta,\phi)| = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}\, e^{-i\frac{\phi}{2}} \end{pmatrix}$$

$$|\sigma'_{\theta,\phi} = -1\rangle\langle\sigma'_{\theta,\phi} = -1| = \frac{1}{2}\left[1 - \vec{\sigma}\cdot\hat{n}\right]$$

13 ## Unitary operators

$A = \sigma_z$ $\qquad \sigma_z' = a_1, a_2 \qquad a_1 = +1, \; a_2 = -1$

$B = \sigma_x$ $\qquad \sigma_x' = b_1, b_2 \qquad b_1 = +1, \; b_2 = -1$

Define:

$$U_{ab} = \sum_{k=1}^{2} |a_k\rangle\langle b_k|$$

$$U_{ba} = \sum_{k=1}^{2} |b_k\rangle\langle a_k|$$

$$\left(\langle a_k|\right)^\dagger = |a_k\rangle$$

14 We can verify the following:

(i) $$U_{ab}^\dagger = \left(\sum_k |a_k\rangle\langle b_k|\right)^\dagger = \sum_k |b_k\rangle\langle a_k| = U_{ba}.$$

(ii) $$U_{ab}U_{ba} = \sum_i |a_i\rangle\langle b_i| \sum |b_j\rangle\langle a_j|$$

$$= \sum_i \sum_j |a_i\rangle \underbrace{\langle b_i|b_j\rangle}_{\delta_{ij}} \langle a_j|$$

$$= \sum_i |a_i\rangle\langle a_i| = 1$$

(iii) Thus, $$U_{ab} = (U_{ba})^{-1} = (U_{ba})^\dagger$$

15 In general

$$U_{ab}|b_k\rangle = |a_k\rangle$$

$$\langle a_k| U_{ab} = \langle b_k|$$

$$U_{ba}|a_k\rangle = |b_k\rangle$$

$$\langle b_k| U_{ba} = \langle a_k|$$

16 For example,

$$\langle \sigma_z' | \; U_{\sigma_z' \sigma_x'} = \langle \sigma_x' |$$

where

$$U_{\sigma_z' \sigma_x'} = | \sigma_z' = +1 \rangle \langle \sigma_x' = +1 | + | \sigma_z' = -1 \rangle \langle \sigma_x' = -1 |$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ -1 & +1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi/2}{2}\right) & \sin\left(\frac{\pi/2}{2}\right) \\ -\sin\left(\frac{\pi/2}{2}\right) & \cos\left(\frac{\pi/2}{2}\right) \end{pmatrix}$$

17 A more general example is

$$\langle \sigma'_{\theta_1, \phi_1} | \; U_{\sigma' \sigma''}(\theta_1, \phi_1, \theta_2, \phi_2) = \langle \sigma''_{\theta_2, \phi_2} |$$

where

$$U_{\sigma'\sigma''}(\theta_1, \phi_1, \theta_2, \phi_2) = | \sigma'_{\theta_1, \phi_1} = +1 \rangle \langle \sigma''_{\theta_2, \phi_2} = +1 | + | \sigma'_{\theta_1, \phi_1} = -1 \rangle \langle \sigma''_{\theta_2, \phi_2} = -1 |$$

(18) $U_{\sigma'\sigma''}(\theta_1, \phi_1, \theta_2, \phi_2)$

$$= |\sigma'_{\theta_1,\phi_1} = +1\rangle\langle\sigma''_{\theta_2,\phi_2} = +1| + |\sigma'_{\theta_1,\phi_1} = -1\rangle\langle\sigma''_{\theta_2,\phi_2} = -1|$$

$$= \begin{pmatrix} \cos\frac{\theta_1}{2} \, e^{-i\frac{\phi_1}{2}} \\ \sin\frac{\theta_1}{2} \, e^{i\frac{\phi_1}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta_2}{2} \, e^{i\frac{\phi_2}{2}} & \sin\frac{\theta_2}{2} \, e^{-i\frac{\phi_2}{2}} \end{pmatrix} + \begin{pmatrix} -\sin\frac{\theta_1}{2} \, e^{-i\frac{\phi_1}{2}} \\ \cos\frac{\theta_1}{2} \, e^{i\frac{\phi_1}{2}} \end{pmatrix} \begin{pmatrix} -\sin\frac{\theta_2}{2} \, e^{i\frac{\phi_2}{2}} & \cos\frac{\theta_2}{2} \, e^{-i\frac{\phi_2}{2}} \end{pmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \, e^{-\frac{i}{2}(\phi_1-\phi_2)} & \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \, e^{-\frac{i}{2}(\phi_1+\phi_2)} \\ \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \, e^{\frac{i}{2}(\phi_1+\phi_2)} & \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \, e^{\frac{i}{2}(\phi_1-\phi_2)} \end{bmatrix} + \begin{bmatrix} \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \, e^{-\frac{i}{2}(\phi_1-\phi_2)} & -\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \, e^{-\frac{i}{2}(\phi_1+\phi_2)} \\ -\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \, e^{\frac{i}{2}(\phi_1+\phi_2)} & \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} \, e^{\frac{i}{2}(\phi_1-\phi_2)} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(\frac{\theta_1-\theta_2}{2}\right) e^{-\frac{i}{2}(\phi_1-\phi_2)} & -\sin\left(\frac{\theta_1-\theta_2}{2}\right) e^{-\frac{i}{2}(\phi_1+\phi_2)} \\ \sin\left(\frac{\theta_1-\theta_2}{2}\right) e^{\frac{i}{2}(\phi_1+\phi_2)} & \cos\left(\frac{\theta_1-\theta_2}{2}\right) e^{\frac{i}{2}(\phi_1-\phi_2)} \end{bmatrix}$$

Interpret $(\phi_1 + \phi_2)$?

eigen vectors.

This seems to be the diagonalization matrix, since they are constructed out of eigen vectors. Set $\phi_2 = 0$.

(19) Let us find the diagonalizat matrix of $\vec{\sigma}\cdot\hat{n}$ in another way.

$$\vec{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$= \sigma_z \cos\theta + \sigma_x \sin\theta \cos\phi + \sigma_y \sin\theta \sin\phi$$

$$= \sigma_z \cos\theta + \sin\theta \left(\sigma_x \cos\phi + \sigma_y \sin\phi \right)$$

(20) Using $\sigma_y = i\,\sigma_x \sigma_z = -i\,\sigma_z \sigma_x$

we have.

$$\sigma_x \cos\phi + \sigma_y \sin\phi = \sigma_x \cos\phi + i\,\sigma_x \sigma_z \sin\phi = \cos\phi\, \sigma_x - i \sin\phi\, \sigma_z \sigma_x$$

$$= \sigma_x \left(\cos\phi + i\,\sigma_z \sin\phi \right) = \left(\cos\phi - i\,\sigma_z \sin\phi \right) \sigma_x$$

$$= \sigma_x\, e^{i\phi\sigma_z} = e^{-i\phi\sigma_z}\, \sigma_x \quad \text{— (i)}$$

$$= \sigma_x\, e^{i\frac{\phi}{2}\sigma_z}\, e^{i\frac{\phi}{2}\sigma_z}$$

$$= e^{-\frac{i}{2}\phi\sigma_z}\, \sigma_x\, e^{\frac{i}{2}\phi\sigma_z} \quad (\text{using (i)})$$

21 Using 20 in 19 we have.

$$\vec{\sigma} \cdot \hat{n} = \sigma_z \cos\theta + \sin\theta \, e^{-\frac{i}{2}\phi\sigma_z} \sigma_x \, e^{\frac{i}{2}\phi\sigma_z}$$

$$= e^{-\frac{i}{2}\phi\sigma_z} \left[\sigma_z \cos\theta + \sigma_x \sin\theta \right] e^{\frac{i}{2}\phi\sigma_z} .$$

22 Show that, like in 25,

$$\sigma_z \cos\theta + \sigma_x \sin\theta = e^{-\frac{i}{2}\theta\sigma_y} \, \sigma_z \, e^{\frac{i}{2}\theta\sigma_y}$$

23 Using 22 in 21

$$\vec{\sigma} \cdot \hat{n} = e^{-\frac{i}{2}\phi\sigma_z} \, e^{-\frac{i}{2}\theta\sigma_y} \, \sigma_z \, \underbrace{e^{\frac{i}{2}\theta\sigma_y} \, e^{\frac{i}{2}\phi\sigma_z}}_{U_{\sigma_z', \sigma''_{\theta,\phi}}}$$

24

$$U_{\sigma'_{\theta,\phi}, \sigma''_{0,0}} = e^{-\frac{i}{2}\phi\sigma_z} \, e^{-\frac{i}{2}\theta\sigma_y}$$

$$= \left(\cos\frac{\phi}{2} - i\sigma_z \sin\frac{\phi}{2} \right) \left(\cos\frac{\theta}{2} - i\sigma_y \sin\frac{\theta}{2} \right)$$

$$= \begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} & -\sin\frac{\theta}{2} \, e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} \, e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2} \, e^{i\frac{\phi}{2}} \end{pmatrix}$$

which matches 18 for $\phi_2 = 0$ there.