Complementary variables: Position and momentum

① In the Stern-Gerlach setup we have seen

$$p([+;\theta_1,\phi_1] \to [\pm;\theta_2,\phi_2]) = |\psi_+(\theta_1,\phi_1)\,\psi_\pm(\theta_2,\phi_2)|^2$$

$$= \Big|\sum_{\sigma'} \langle\sigma'|\sigma'_{\theta_1,\phi_1}=+\rangle^\dagger \langle\sigma'|\sigma'_{\theta_2,\phi_2}=\pm 1\rangle\Big|^2$$

$$= \Big|\sum_{\sigma'} \langle\sigma'_{\theta_1,\phi_1}=+1|\sigma'\rangle\langle\sigma'|\sigma'_{\theta_2,\phi_2}=\pm 1\rangle\Big|^2$$

$$= |\langle\sigma'_{\theta_1,\phi_1}=+1|\sigma'_{\theta_2,\phi_2}=\pm 1\rangle|^2$$

$$= \langle\sigma'_{\theta_2,\phi_2}=\pm 1|\sigma'_{\theta_1,\phi_1}=+1\rangle\langle\sigma'_{\theta_1,\phi_1}=+1|\sigma'_{\theta_2,\phi_2}=\pm 1\rangle$$

$$= tr\, |\sigma'_{\theta_1,\phi_1}=+1\rangle\langle\sigma'_{\theta_1,\phi_1}=+1|\sigma'_{\theta_2,\phi_2}=\pm 1\rangle\langle\sigma'_{\theta_2,\phi_2}=\pm 1|$$

$$= tr\left(\frac{1+\vec{\sigma}\cdot\hat{n}(\theta_1,\phi_1)}{2}\right)\left(\frac{1\pm\vec{\sigma}\cdot\hat{n}(\theta_2,\phi_2)}{2}\right)$$

$$\big\downarrow \text{HW}$$

$$= \frac{1\pm\cos\Theta}{2}$$

where

$$\sigma_{\theta,\phi} = \vec{\sigma}\cdot\hat{n}(\theta,\phi)$$

$$\cos\Theta = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)$$

(2) A particular case of interest here is

$$p\left([+;\tfrac{\pi}{2},0] \to [\pm;\tfrac{\pi}{2},\tfrac{\pi}{2}]\right) = \left|\langle \sigma_x'=+1 | \sigma_y'=\pm1\rangle\right|^2$$

$$= \langle \sigma_y'=\pm1 | \sigma_x'=+1\rangle \langle \sigma_x'=+1 | \sigma_y'=\pm1\rangle$$

$$= tr\ |\sigma_x'=+1\rangle\langle\sigma_x'=+1|\sigma_y'=\pm1\rangle\langle\sigma_y'=\pm1|$$

$$= tr\left(\frac{1+\sigma_x}{2}\right)\left(\frac{1\pm\sigma_y}{2}\right)$$

$$= \frac{1}{4}\, tr\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & \mp i\\ \pm i & 1\end{pmatrix}$$

$$= \frac{1}{4}\, tr\begin{pmatrix}1\pm i & 1\mp i\\ 1\pm i & 1\mp i\end{pmatrix} = \frac{1}{2}$$

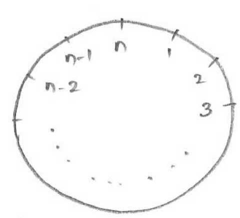
(3) Now consider a vector space spanned by

$$|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle,$$

which form an orthonormal basis set,

$$\langle a_l | a_m\rangle = \delta_{lm}.$$

(4) Consider a unitary operator U that cyclically permutes the eigenvectors,



$$U|a_k\rangle = |a_{k+1}\rangle$$

for $k = 1, 2, \ldots, n-1$, and

$$U|a_n\rangle = |a_1\rangle$$

5 Repeating the operations we obtain

$$U \mid a_k \rangle = \mid a_{k+1} \rangle$$

$$U^2 \mid a_k \rangle = \mid a_{k+2} \rangle$$

$$\vdots$$

$$U^n \mid a_k \rangle = \mid a_k \rangle$$

6 Thus, $U^n = 1$.

7 Let us find the eigenvalues u'_k and eigenvectors $\mid u'_k \rangle$ of the operator U.

8 The eigenvalues satisfy

$$(u')^n = 1,$$

which implies the eigenvalues are the n-th roots of unity,

$$u'_k = e^{i \frac{2\pi}{n} k}, \qquad k = 1, 2, \ldots, n.$$

9 We now want to represent the operator

$$\mid u'_k \rangle \langle u'_k \mid$$

in terms of U, analogous to (in $n=2$ case)

$$\mid \sigma_x' = +1 \rangle \langle \sigma_x' = +1 \mid = \frac{1 + \sigma_x}{2}$$

and

$$\mid \sigma_x' = -1 \rangle \langle \sigma_x' = -1 \mid = \frac{1 - \sigma_x}{2}.$$

10 Note that

$$U^n - 1 = 0$$

$$U^n - 1 = (U-1)(1 + U + U^2 + \dots + U^{n-1})$$

$$U^n - 1 = (U - u_1')(U - u_2') \dots (U - u_n')$$

11 The construction is

$$|u_k'\rangle\langle u_k'| = \text{constant} \frac{(U-u_1')(U-u_2')\dots(U-u_n')}{(U-u_k')}$$

$$= \text{constant} \frac{(U^n - 1)}{(U - u_k')}$$

$$= \text{constant} \frac{\left(\left(\frac{U}{u_k'}\right)^n - 1\right)}{u_k'\left(\frac{U}{u_k'} - 1\right)}$$

$$\left(\frac{U}{u_k'}\right)^n = U^n e^{-i\frac{2\pi}{n}kn} = U^n e^{-i2\pi k} = U^n .$$

$$= \text{constant} \frac{1}{u_k'} \sum_{l=0}^{n-1} \left(\frac{U}{u_k'}\right)^l$$

$$= \text{constant} \frac{1}{u_k'} \sum_{l=1}^{n} \left(\frac{U}{u_k'}\right)^l \qquad \because \left(\frac{U}{u_k'}\right)^n = 1$$

12 To determine the constant we multiply by $|u_k'\rangle$

$$|u_k'\rangle\underbrace{\langle u_k' | u_k'\rangle}_{=1} = \text{constant} \frac{1}{u_k'} \sum_{l=1}^{n} \left(\frac{U}{u_k'}\right)^l |u_k'\rangle$$

$$U|u_k'\rangle = u_k'|u_k'\rangle$$

$$= \text{constant} \frac{1}{u_k'} \sum_{l=1}^{n} |u_k'\rangle$$

$$= \text{constant} \frac{1}{u_k'} \, n \, |u_k'\rangle$$

$$\Rightarrow \text{constant} = \frac{u_k'}{n}$$

(13) Using (12) in (11) we have

$$| u'_k \rangle \langle u'_k | = \frac{1}{n} \sum_{l=1}^{n} \left(\frac{U}{u'_k} \right)^l$$

(14) We verify that for $n=2$, letting $U = \sigma_x$, with $u'_1 = +1$, $u'_2 = -1$,

$$| u'_1 \rangle \langle u'_1 | = \frac{1}{2} \sum_{l=1}^{2} \left(\frac{\sigma_x}{u'_1} \right)^2$$

$$|\sigma_x' = +1\rangle \langle \sigma_x' = +1| = \frac{1}{2} \left[\frac{\sigma_x}{+1} + \left(\frac{\sigma_x}{+1} \right)^2 \right] = \frac{1}{2} (1 + \sigma_x) \quad \checkmark$$

$$| u'_2 \rangle \langle u'_2 | = \frac{1}{2} \sum_{l=1}^{2} \left(\frac{\sigma_x}{-1} \right)^2$$

$$|\sigma_x' = -1\rangle \langle \sigma_x' = -1| = \frac{1}{2} \left[\frac{\sigma_x}{-1} + \left(\frac{\sigma_x}{-1} \right)^2 \right] = \frac{1}{2} (1 - \sigma_x) \quad \checkmark$$

(15) Multiplying (13) by $| a_n \rangle$ we obtain

$$| u'_k \rangle \langle u'_k | a_n \rangle = \frac{1}{n} \sum_{l=1}^{n} \left(\frac{U}{u'_k} \right)^l | a_n \rangle$$

$$= \frac{1}{n} \sum_{l=1}^{n} \frac{1}{(u'_k)^l} | a_{n+l} \rangle$$

$$= \frac{1}{n} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} k l} | a_l \rangle$$

(16) Now, multiplying by $\langle a_n |$ from left.

$$\langle a_n | u'_k \rangle \langle u'_k | a_n \rangle = \frac{1}{n} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} k l} \underbrace{\langle a_n | a_l \rangle}_{\delta_{nl}}$$

$$= \frac{1}{n} e^{-i \frac{2\pi}{n} k n}$$

$$= \frac{1}{n}$$

(17) Thus,

$$\left| \langle u_k' | a_n \rangle \right|^2 = \frac{1}{n}$$

$$\Rightarrow \langle u_k' | a_n \rangle = \frac{1}{\sqrt{n}}$$, choosing phase to be zero.

(18) Using (17) in (15) we have.

$$|u_k'\rangle \frac{1}{\sqrt{n}} = \frac{1}{n} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} |a_l\rangle$$

$$|u_k'\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} |a_l\rangle$$ → compare with Fourier transform.

(19) Multiplying by $\langle a_m|$ from left we have the direction cosines

$$\langle a_m | u_k' \rangle = \frac{1}{\sqrt{n}} e^{-i\frac{2\pi}{n}km}.$$

(20) Observe that

$$\delta_{lm} = \langle a_l | a_m \rangle = \sum_{k=1}^{n} \langle a_l | u_k' \rangle \langle u_k' | a_m \rangle$$

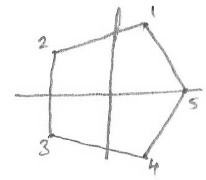
$$= \frac{1}{n} \sum_{k=1}^{n} e^{i\frac{2\pi}{n}k(m-l)}$$

Choose $l = n$, then

$$m \neq n \Rightarrow \sum_{k=1}^{n} e^{i\frac{2\pi}{n}km} = 0$$

⇒ all m-th powers of n-th roots of unity sum up to zero.

(21) $$\sum_{k=1}^{n} e^{i\frac{2\pi}{n}km} = 0 \qquad \text{for } m \neq n.$$



Let $\underline{\underline{n=5}}$

$\underline{m=1}$: $$e^{i\frac{2\pi}{5}} + e^{i\frac{2\pi}{5}2} + e^{i\frac{2\pi}{5}3} + e^{i\frac{2\pi}{5}4} + 1 = 0$$

$\underline{m=2}$: $$e^{i\frac{2\pi}{5}2} + e^{i\frac{2\pi}{5}4} + e^{i\frac{2\pi}{5}6} + e^{i\frac{2\pi}{5}8} + e^{i\frac{2\pi}{5}10}$$
$$= e^{i\frac{2\pi}{5}2} + e^{i\frac{2\pi}{5}4} + e^{i\frac{2\pi}{5}1} + e^{i\frac{2\pi}{5}3} + 1 = 0$$

$\underline{m=3}$: $$e^{i\frac{2\pi}{5}3} + e^{i\frac{2\pi}{5}6} + e^{i\frac{2\pi}{5}9} + e^{i\frac{2\pi}{5}12} + e^{i\frac{2\pi}{5}15}$$
$$= e^{i\frac{2\pi}{5}3} + e^{i\frac{2\pi}{5}1} + e^{i\frac{2\pi}{5}4} + e^{i\frac{2\pi}{5}2} + 1 = 0$$

$\underline{m=4}$: $$e^{i\frac{2\pi}{5}4} + e^{i\frac{2\pi}{5}8} + e^{i\frac{2\pi}{5}12} + e^{i\frac{2\pi}{5}16} + e^{i\frac{2\pi}{5}20}$$
$$= e^{i\frac{2\pi}{5}4} + e^{i\frac{2\pi}{5}3} + e^{i\frac{2\pi}{5}2} + e^{i\frac{2\pi}{5}1} + 1 = 0$$

(22) Summary:

→ $|a_k\rangle \qquad k = 1, 2, \ldots, n$

→ $U|a_k\rangle = |a_{k+1}\rangle, \quad U|a_n\rangle = |a_1\rangle, \quad U^{\dagger} = U, \Rightarrow U^n = 1.$

→ Eigenvectors of U: $|u'_k\rangle, \quad u'_k = e^{i\frac{2\pi}{n}k}.$

→ $$|u'_k\rangle\langle u'_k| = \frac{1}{n}\sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl}\, U^l$$

→ $$|u'_k\rangle = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl}\,|a_l\rangle$$

→ $$\langle a_m|u'_k\rangle = \frac{1}{\sqrt{n}}\, e^{-i\frac{2\pi}{n}km}$$

(23) Let us next consider the operation

$$\langle u'_k | V = \langle u'_{k+1} |$$

such that

$$\langle u'_n | V = \langle u'_1 | .$$

(24) Now, $V^n = 1$, and we have.

$$V'_k = e^{i \frac{2\pi}{n} k} \quad \text{— eigenvalues.}$$

$$\langle V'_k | \quad \text{— eigenvectors.}$$

(25) Similar to the earlier case we obtain

$$| V'_k \rangle \langle V'_k | = \frac{1}{n} \sum_{l=1}^{n} \left(\frac{V}{V'_k} \right)^l$$

and

$$\langle u'_n | V'_k \rangle \langle V'_k | = \frac{1}{n} \sum_{l=1}^{n} \frac{1}{(V'_k)^l} \langle u'_n | V^l .$$

$$= \frac{1}{n} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \langle u'_l |$$

$$\langle u'_n | V'_k \rangle \langle V'_k | u'_n \rangle = \frac{1}{n}$$

$$\Rightarrow \quad \langle u'_n | V'_k \rangle = \frac{1}{\sqrt{n}} \quad \text{(choosing phase to be zero.)}$$

Thus,

$$\langle V'_k | = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \langle u'_l |$$

26 But, using adjoint of 18,

$$\langle U'_l | = \frac{1}{\sqrt{n}} \sum_{m=1}^{n} e^{i \frac{2\pi}{n} lm} \langle a_m |$$

27 Using 26 in 25

$$\langle V'_k | = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \frac{1}{\sqrt{n}} \sum_{m=1}^{n} e^{i \frac{2\pi}{n} lm} \langle a_m |$$

$$= \frac{1}{n} \sum_{m=1}^{n} \langle a_m | \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} l(k-m)}$$

$$= \sum_{m=1}^{n} \langle a_m | \delta_{km} \qquad \because \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} l(k-m)} = n\, \delta_{km}$$

see 20, 21.

$$= \langle a_k | .$$

28 <u>Summary</u>.

$\rightarrow$ $U^\dagger U = 1$, $U^n = 1$, $|U'_k\rangle$, $U'_k = e^{i \frac{2\pi}{n} k}$

$V^\dagger V = 1$, $V^n = 1$, $|V'_k\rangle$, $V'_k = e^{i \frac{2\pi}{n} k}$

$\rightarrow$ $U \, |V'_k\rangle = |V'_{k+1}\rangle$

$V^\dagger \, |U'_k\rangle = |U'_{k+1}\rangle$

$$\rightarrow \quad |U'_k\rangle \langle U'_k| = \frac{1}{n} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \, U^l$$

$$|V'_k\rangle \langle V'_k| = \frac{1}{n} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \, V^l .$$

$$\rightarrow \quad |U'_k\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i \frac{2\pi}{n} kl} \, |V'_l\rangle$$

$$|V'_l\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{i \frac{2\pi}{n} kl} \, |U'_k\rangle$$