# Position and momentum

SET-10

1. There exists $|v'_k\rangle$ $\quad k = 1, 2, \ldots, n$

**Define U:** $\quad U^\dagger U = 1$ $\quad$ and $\quad U|v'_k\rangle = |v'_{k+1}\rangle$ $\quad \Rightarrow \quad U^n = 1$

$U|v'_n\rangle = |v'_1\rangle$

(i) Define $|u'_k\rangle$ as eigenstates of $U$: $\quad U|u'_k\rangle = u'_k|u'_k\rangle$

(ii) Eigenvalues: $\quad u'_k = e^{i\frac{2\pi}{n}k}$ $\quad$ verify this

(iii) Eigenvectors: $\quad |u'_k\rangle = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl}|v'_l\rangle$

(iv) Projection op.: $\quad |u'_k\rangle\langle u'_k| = \frac{1}{n}\sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl}U^l$ $\quad \rightarrow \hat{i}\hat{i} = \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$

(v) Completeness: $\quad \sum_{k=1}^{n}|u'_k\rangle\langle u'_k| = 1$

(vi) Transformation function: $\quad \langle v'_m|u'_k\rangle = \frac{1}{\sqrt{n}}e^{-i\frac{2\pi}{n}km}$

**Define V:** $\quad V^\dagger V = 1$ $\quad$ and $\quad V^\dagger|u'_k\rangle = |u'_{k+1}\rangle$ $\quad \Rightarrow \quad V^n = 1$

$V^\dagger|u'_n\rangle = |u'_1\rangle$

(i) Define $|v'_k\rangle$ as eigenstates of $V^\dagger$: $\quad V^\dagger|v'_k\rangle = v'_k|v'_k\rangle$

(ii) Eigenvalues: $\quad v'_k = e^{i\frac{2\pi}{n}k}$

(iii) Eigenvectors: $\quad |v'_k\rangle = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} e^{i\frac{2\pi}{n}kl}|u'_l\rangle$

(iv) Projection op.: $\quad |v'_k\rangle\langle v'_k| = \frac{1}{n}\sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl}V^l$

(v) Completeness relation: $\quad \sum_{k=1}^{n}|v'_k\rangle\langle v'_k| = 1$

(vi) Transformation function: $\quad \langle u'_m|v'_k\rangle = \frac{1}{\sqrt{n}}e^{i\frac{2\pi}{n}km}$

2 Let $n = 2$.

Let $|v_1'\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |v_2'\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

<u>Define</u> $U = \sigma_x$ :

$$\sigma_x^{+}\sigma_x = 1$$

$$\sigma_x |v_1'\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |v_2'\rangle$$

$$\sigma_x |v_2'\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |v_1'\rangle$$

$$\sigma_x^2 = 1$$

(i) Let $|u_k'\rangle$ be the eigenvalues of $\sigma_x$.

(ii) $u_k' = e^{i\frac{2\pi}{n}k}$ ; $\quad u_1' = -1$ , $u_2' = +1$

(iii) $$|u_k'\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} |v_l'\rangle$$

$$|u_1'\rangle = \frac{1}{\sqrt{2}}\left[ (-1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)^2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \psi_-\left(\frac{\pi}{2}, 0\right)$$

$$|u_2'\rangle = \frac{1}{\sqrt{2}}\left[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \psi_+\left(\frac{\pi}{2}, 0\right)$$

(iv) $$|u_k'\rangle\langle u_k'| = \frac{1}{n} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} U^l.$$

$$|u_1'\rangle\langle u_1'| = \frac{1}{2}\left[ (-1)\sigma_x + (-1)^2\sigma_x^2 \right] = \frac{1-\sigma_x}{2}$$

$$|u_2'\rangle\langle u_2'| = \frac{1}{2}\left[ \sigma_x + \sigma_x^2 \right] = \frac{1+\sigma_x}{2}$$

(v) $$\sum_{k=1}^{n} |u_k'\rangle\langle u_k'| = 1 : \quad \frac{1-\sigma_x}{2} + \frac{1+\sigma_x}{2} = 1$$

(vi) $$\langle v_m' | u_k'\rangle = \frac{1}{\sqrt{n}} e^{-i\frac{2\pi}{n}km}$$

$$\langle v_1' | u_1'\rangle = (1 \;\; 0)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \qquad \langle v_1' | u_2'\rangle = (1 \;\; 0)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\langle v_2' | u_1'\rangle = (0 \;\; 1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \qquad \langle v_2' | u_2'\rangle = (0 \;\; 1)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

(3) Define $V = \sigma_z$ :

$$\sigma_z^\dagger \sigma_z = 1$$

arbitrary phase.

$$\sigma_z^\dagger | u_1' \rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -|u_2'\rangle$$

$$\sigma_z^\dagger | u_2' \rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -|u_1'\rangle$$

$$\sigma_z^2 = 1$$

(i) Let $|v_k'\rangle$ be eigenstates of $\sigma_z^\dagger$

(ii) $v_k' = e^{i\frac{2\pi}{n}k}$ : $\quad v_1' = -1, \quad v_2' = +1$

(iii) $|v_k'\rangle = \frac{1}{\sqrt{n}} \sum_{l=1}^{n} e^{i\frac{2\pi}{n}kl} |u_l'\rangle$

$$|v_1'\rangle = \frac{1}{\sqrt{2}} \left[ (-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} + (-1)^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|v_2'\rangle = \frac{1}{\sqrt{2}} \left[ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \right] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(iv) $|v_k'\rangle\langle v_k'| = \frac{1}{n} \sum_{l=1}^{n} e^{-i\frac{2\pi}{n}kl} V^l$

$$|v_1'\rangle\langle v_1'| = \frac{1}{2} \left[ (-1)\sigma_z + (-1)^2 \sigma_z^2 \right] = \frac{1-\sigma_z}{2}$$

$$|v_2'\rangle\langle v_2'| = \frac{1}{2} \left[ \sigma_z + \sigma_z^2 \right] = \frac{1+\sigma_z}{2}$$

(v) $\sum_{k=1}^{n} |v_k'\rangle\langle v_k'| = 1$ : $\quad \frac{1-\sigma_z}{2} + \frac{1+\sigma_z}{2} = 1.$

(vi) $\langle u_m' | v_k' \rangle = \frac{1}{\sqrt{n}} e^{i\frac{2\pi}{n}km}$

$$\langle u_1' | v_1' \rangle = \frac{1}{\sqrt{2}} (-1 \;\; 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{2}} \qquad \langle u_1' | v_2' \rangle = \frac{1}{\sqrt{2}} (-1 \;\; 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

$$\langle u_2' | v_1' \rangle = \frac{1}{\sqrt{2}} (1 \;\; 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \qquad \langle u_2' | v_2' \rangle = \frac{1}{\sqrt{2}} (1 \;\; 1) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}$$

4 For $n=2$ we have.

$$p\left([\sigma_x=+1] \to [\sigma_x=+1]\right) = tr\left(\frac{1+\sigma_x}{2}\right)\left(\frac{1+\sigma_x}{2}\right) = 1$$

$$p\left([\sigma_x=+1] \to [\sigma_z=+1] \to [\sigma_x=+1]\right)$$

$$= tr\left(\frac{1+\sigma_x}{2}\right)\left(\frac{1+\sigma_z}{2}\right)\left(\frac{1+\sigma_x}{2}\right)$$

$$= \frac{1}{8}\, tr\left(1+\sigma_x+\sigma_y+\sigma_z+\sigma_x\sigma_z+\sigma_x^2+\sigma_z\sigma_x+\sigma_x\sigma_z\sigma_x\right)$$

$$= \frac{1}{2}$$

$$p\left([\sigma_z=+1] \to [\sigma_x=+1]\right) = tr\left(\frac{1+\sigma_z}{2}\right)\left(\frac{1+\sigma_x}{2}\right)$$

$$= \frac{1}{4}\, tr\left(1+\sigma_z+\sigma_x+\sigma_z\sigma_x\right)$$

$$= \frac{1}{2}$$

Thus, measurement of $\sigma_z=+1$ completely erases the previous measurement of $\sigma_x=+1$.

5 What is generalization of this for higher $n$?

$$p([u'_k] \to [u'_k]) = tr\, |u'_k\rangle\langle u'_k|u'_k\rangle\langle u'_k|$$

$$= 1$$

$$p([u'_k] \to [v'_m] \to [u'_k])$$

$$= tr\, |u'_k\rangle\langle u'_k|v'_m\rangle\langle v'_m|u'_k\rangle\langle u'_k|$$

$$= \langle u'_k|v'_m\rangle\langle v'_m|u'_k\rangle \underbrace{\langle u'_k|u'_k\rangle}_{=1}$$

$$= |\langle v'_m|u'_k\rangle|^2$$

$$= \left|\frac{1}{\sqrt{n}} e^{-i\frac{2\pi}{n}km}\right|^2 \quad \text{(using 1)}$$

$$= \frac{1}{n}$$

$$p([v'_m] \to [u'_k]) = tr\, |v'_m\rangle\langle v'_m|u'_k\rangle\langle u'_k|$$

$$= \langle v'_m|u'_k\rangle\langle u'_k|v'_m\rangle$$

$$= |\langle u'_k|v'_m\rangle|^2$$

$$= \frac{1}{n}$$

Thus, measurement of $[v'_m]$ completely erases the previous measurement of $[u'_k]$. Thus, $U$ and $V$ are maximally incompatible. $U$ and $V$ are called complementary variables.

(6) For $n=2$ we can construct four operators

$$\begin{matrix} UV & UV^2 \\ U^2V & U^2V^2 \end{matrix}$$

For $U=\sigma_x$ and $V=\sigma_z$ these are.

$$UV = \sigma_x \sigma_z = -i\sigma_y$$

$$UV^2 = \sigma_x$$

$$U^2V = \sigma_z$$

$$U^2V^2 = 1$$

Taking the trace we have.

$$tr(UV) = tr(-i\sigma_y) = 0$$

$$tr(UV^2) = tr(\sigma_x) = 0$$

$$tr(U^2V) = tr(\sigma_z) = 0$$

$$tr(U^2V^2) = tr(1) = 2$$

(7) In general we have $n^2$ independent operators

$$\begin{matrix} UV & UV^2 & \cdots & UV^n \\ U^2V & U^2V^2 & \cdots & U^2V^n \\ \vdots & \vdots & \ddots & \vdots \\ U^nV & U^nV^2 & \cdots & U^nV^n \end{matrix}$$

8

$$\operatorname{tr}(U^k V^m) = \operatorname{tr}\left[\left(\sum_{l=1}^{n} |u'_l\rangle\langle u'_l|\right)(U^k V^m)\right]$$

$$= \sum_{l=1}^{n} \operatorname{tr}\left(|u'_l\rangle\langle u'_l| U^k V^m\right)$$

$$= \sum_{l=1}^{n} \langle u'_l | U^k V^m | u'_l\rangle$$

$$= \sum_{l=1}^{n} \langle u'_l | U^k | u'_{l+m}\rangle$$

$$= \sum_{l=1}^{n} e^{i\frac{2\pi}{n}(l+m)k} \underbrace{\langle u'_l | u'_{l+m}\rangle}_{\delta_{mn}}$$

$$= \sum_{l=1}^{n} e^{i\frac{2\pi}{n}(l+m)k} \delta_{mn}$$

$$= \delta_{mn} \underbrace{e^{i\frac{2\pi}{n}mk}}_{=1} \sum_{l=1}^{n} e^{i\frac{2\pi}{n}lk}$$

$$= \delta_{mn}\, \delta_{kn}\, n$$

$$= \begin{cases} 0 & \text{if } m \neq n \text{ and } k \neq n \\ n & \text{if } m = n \text{ and } k = n \end{cases}$$

(9) $$\operatorname{tr} F(U,V) = \operatorname{tr} \sum_{l=1}^{n} \sum_{m=1}^{n} f_{lm} \, U^l V^m$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} f_{lm} \operatorname{tr} U^l V^m$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} f_{lm} \, \delta_{ln} \, \delta_{mn} \, n$$

$$= n f_{nn}$$

$$= n f_{00}$$

$$\sum_{l=1}^{n} \sum_{m=1}^{n} F(u'_l, v'_m) = \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{l'=1}^{n} \sum_{m'=1}^{n} f_{l'm'} (u'_l)^{l'} (v'_m)^{m'}$$

$$= \sum_{l=1}^{n} \sum_{m=1}^{n} \sum_{l'=1}^{n} \sum_{m'=1}^{n} f_{l'm'} \, e^{i \frac{2\pi}{n} ll'} \, e^{i \frac{2\pi}{n} mm'}$$

$$= \sum_{l'=1}^{n} \sum_{m'=1}^{n} f_{l'm'} \underbrace{\sum_{l=1}^{n} e^{i \frac{2\pi}{n} ll'}}_{n \, \delta_{l'n}} \underbrace{\sum_{m=1}^{n} e^{i \frac{2\pi}{n} mm'}}_{n \, \delta_{m'n}}$$

$$= n^2 f_{nn}$$

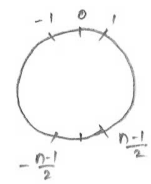
$$= n \operatorname{tr} F(U,V)$$

Thus, we have.

$$\operatorname{tr} F(U,V) = \frac{1}{n} \sum_{l=1}^{n} \sum_{m=1}^{n} F(u'_l, v'_m)$$ $\longrightarrow$ Ergodic theorem

10 Let $n$ be odd.

$$k = \frac{n-1}{2}, \ldots, 0, \ldots, -\frac{n-1}{2}$$




$$\frac{2\pi}{n} = \epsilon^2$$

We will take the limit $n \to \infty$ and $\epsilon \to 0$.

11
$$U = e^{i\epsilon Q} \qquad u'_k = e^{i\frac{2\pi}{n}k} = e^{i\epsilon q'} \qquad q' = \epsilon k$$

$$V = e^{i\epsilon P} \qquad V'_k = e^{i\frac{2\pi}{n}k} = e^{i\epsilon p'} \qquad p' = \epsilon k$$

12 The periodicity can be restricted by requiring $q'$ and $p'$ to be finite while $n \to \infty$ and $\epsilon \to 0$.

13
$$\langle u'_m | UV = e^{i\frac{2\pi}{n}m} \langle u'_m | V = e^{i\frac{2\pi}{n}m} \langle u'_{m+1} |$$

$$\langle u'_m | VU = \langle u'_{m+1} | U = e^{i\frac{2\pi}{n}(m+1)} \langle u'_{m+1} |$$

$$VU - e^{i\frac{2\pi}{n}} UV = 0$$

$n = 2$ : $VU + UV = 0 \rightarrow$ Stern-Gerlach

$n = 3$ : $VU - e^{i\frac{2\pi}{3}} UV = 0$

$\vdots$

$n \to \infty$ : $VU - UV = 0 \rightarrow$ classical

(14) $\langle u'_m | U^k V^l = e^{i\frac{2\pi}{n}km} \langle u'_m | V^l = e^{i\frac{2\pi}{n}km} \langle u'_{m+l} |$

$\langle u'_m | V^l U^k = \langle u'_{m+l} | U^k = e^{i\frac{2\pi}{n}(m+l)k} \langle u'_{m+l} |$

$V^l U^k = e^{i\frac{2\pi}{n}lk} U^k V^l.$

$V^{-l} U^k V^{+l} = U^k e^{i\frac{2\pi}{n}lk}$

(15) Using (11) in (14)

$\epsilon l = q'$
$\epsilon k = p'$

$e^{-i\epsilon l P} e^{i\epsilon k Q} e^{i\epsilon l P} = e^{i\epsilon k Q} e^{i\epsilon l \epsilon k}$

$e^{-iq' P} e^{ip' Q} e^{iq' P} = e^{ip'(Q-q')}$

$e^{ip'\left(e^{-iq'P} Q e^{iq'P}\right)} = e^{ip'(Q-q')}$ using $e^{-A} f(Q) e^{A} = f(e^{-A} Q e^{A})$

(16) Avoiding periodicity, see (12), for continuous $q'$ and $p'$,

$e^{-iq'P} Q e^{iq'P} = Q - q'$

(17) Taking derivative with respect to $q'$ we obtain

$-iPQ + iQP = -1$

$QP - PQ = i$

18 $$\langle u'_k | V^l = \langle u'_{k+l} |$$

$$\langle q' | e^{iq''P} = \langle q' + q'' |$$

Let $q'' = \delta q'$

$$\langle q' | \left[ 1 + i \delta q' P \right] = \langle q' | + \delta q' \frac{\partial}{\partial q'} \langle q' |$$

$$\langle q' | P | \rangle = \frac{1}{i} \frac{\partial}{\partial q'} \langle q' | \rangle$$

$$\psi(q') = \langle q' | \rangle$$

$$= \frac{1}{i} \frac{\partial}{\partial q'} \psi(q')$$

19 $$U^l | v'_k \rangle = | v'_{k+l} \rangle$$

$$e^{ip''Q} | p' \rangle = | p' + p'' \rangle$$

$$\Rightarrow \quad \langle | Q | p' \rangle = \frac{1}{i} \frac{\partial}{\partial p'} \langle | p' \rangle$$

20 In general

$$\langle q' | f(P, Q) | \rangle = f\left( \frac{1}{i} \frac{\partial}{\partial q'}, q' \right) \langle q' | \rangle$$

$$\langle | f(P, Q) | p' \rangle = f\left( p', \frac{1}{i} \frac{\partial}{\partial p'} \right) \langle | p' \rangle$$