<u>Date</u> : 2014 Sep 3

1 <u>1 - dimension</u>

$$\int_a^b dx \, \frac{d}{dx} f(x) = f(b) - f(a)$$

2 <u>3 - dimensions</u>

(i) $$\int_{\vec{a}}^{\vec{b}} d\vec{l} \cdot \vec{\nabla} f(\vec{x}) = f(\vec{b}) - f(\vec{a})$$

(ii) $$\int_S d\vec{a} \cdot \vec{\nabla} \times \vec{E} = \oint_C d\vec{l} \cdot \vec{E}$$

C - right hand rule.

(iii) $$\int_V d^3x \; \vec{\nabla} \cdot \vec{E} = \oint_S d\vec{a} \cdot \vec{E}$$

3 <u>Example</u>

$$\vec{E} = r^n \hat{r} = r^{n-1} \vec{r}$$

— corresponds to the electric field inside a sphere with charge density

Volume : sphere of radius R

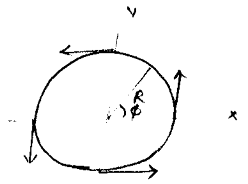
$$\vec{\nabla} \cdot \vec{E} = (n+2) \, r^{n-1}$$

$$\int_V d^3r \; \vec{\nabla} \cdot \vec{E} = 4\pi \, R^{n+2}$$

$$\oint_S d\vec{a} \cdot \vec{E} = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \sin\theta \, r^2 \hat{r} \cdot r^n \hat{r}$$

$$= 4\pi \, R^{n+2}$$

4 <u>Example</u>



$$\vec{A} = -y\,\hat{x} + x\,\hat{y}$$
$$= R\,\hat{\phi}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\vec{\nabla}\times\vec{A} = 2\,\hat{z}$$

$$\int_S d\vec{a}\cdot\vec{\nabla}\times\vec{A} = \pi R^2\,\hat{z}\cdot 2\,\hat{z}$$
$$= 2\pi R^2$$

$$\oint_C d\vec{l}\cdot\vec{A} = \int_0^{2\pi} R\,d\phi\,\hat{\phi}\cdot R\,\hat{\phi}$$
$$= 2\pi R^2$$

To demonstrate that it is independepent of surface, let us integrate on a hemisphere,

$$\int_S d\vec{a}\cdot\vec{\nabla}\times\vec{A} = \int_S d\vec{a}\cdot 2\,\hat{z}$$
$$= 2\,\hat{z}\cdot\int d\vec{a}$$
$$= 2\,\hat{z}\cdot\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^{2\pi}d\phi\;\hat{r}\,R^2$$
$$= 2R^2\,\hat{z}\cdot\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\int_0^{2\pi}d\phi\left[\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}\right]$$
$$= 2R^2\int_0^{\frac{\pi}{2}}\sin\theta\,d\theta\;2\pi\cos\theta$$
$$\int_0^1 dt\,t = \frac{1}{2}$$
$$= 4\pi R^2\underbrace{\int_0^{\pi/2}\sin\theta\,d\theta\,\cos\theta}_{=\frac{1}{2}}$$

$$= 2\pi R^2$$

5 Maxwell's equations in differential form can be derived from their integral counterparts.

(i) $$\oint_S d\vec{a} \cdot \vec{E} = \frac{Q}{\epsilon_0}$$

$$\int_V d^3x \ \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \int d^3x \ \rho \Rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

(ii) $$\oint_S d\vec{a} \cdot \vec{B} = 0$$

$$\int_V d^3x \ \vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

(iii) $$\oint_c d\vec{l} \cdot \vec{E} = -\frac{\partial}{\partial t} \Phi_B$$

$$\int_S d\vec{a} \cdot \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \int_S d\vec{a} \cdot \vec{B} \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(iv) $$\oint_c d\vec{l} \cdot \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \Phi_E + \mu_0 I$$

$$\int_S d\vec{a} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \int_S d\vec{a} \cdot \vec{E} + \mu_0 \int d\vec{a} \cdot \vec{J}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$