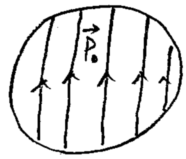
Date : 2014 Oct 27

① Let us consider a solid sphere, of radius R, with permanent polarization

$$\vec{P}(\vec{r},t) = \vec{P}_0 \, \theta(R-r)$$



② Maxwell's equations are.

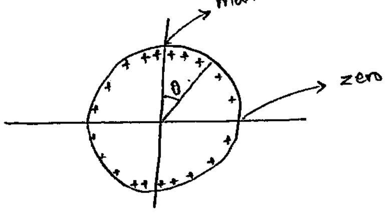
$$\vec{\nabla}\cdot(\epsilon_0 \vec{E}) = \rho \underbrace{- \vec{\nabla}\cdot\vec{P}}_{\rho_{eff.}}$$

$$\vec{\nabla}\times\vec{E} = 0$$

③

$$\begin{aligned}
\rho_{eff}(\vec{r}) &= - \vec{\nabla}\cdot\vec{P} \\
&= - \vec{\nabla}\cdot\left[\vec{P}_0 \, \theta(R-r)\right] \\
&= - \vec{P}_0 \cdot \vec{\nabla} \, \theta(R-r) \\
&= - \left[\vec{P}_0 \cdot (\vec{\nabla} r)\right] \frac{\partial}{\partial r} \theta(R-r) \\
&= (\vec{P}_0 \cdot \hat{r}) \, \delta(R-r) \\
&= P_0 \cos\theta \, \delta(R-r)
\end{aligned}$$

$$\frac{\partial}{\partial x}\theta(R-r) = \frac{\partial r}{\partial x}\frac{\partial}{\partial r}\theta(R$$



(4) To find the electric potential we have.

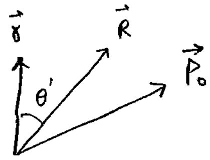
$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \rho_{eff}.$$

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho_{eff}(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^\infty r'^2 dr' \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, \frac{\vec{P}_0 \cdot \left\{ \hat{i} \sin\theta' \cos\phi' + \hat{j} \sin\theta' \sin\phi' + \cos\theta' \hat{k} \right\} \delta(r-}{\sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}}$$

$$= \frac{R^2}{4\pi\epsilon_0} \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, \frac{\vec{P}_0 \cdot \left\{ \hat{i} \sin\theta' \cos\phi' + \hat{j} \sin\theta' \sin\phi' + \cos\theta' \hat{k} \right\}}{\sqrt{r^2 + R^2 - 2\vec{r}\cdot\vec{R}}}$$

(5) Choose $\vec{r}$ to be along z-axis. If we do not make the choice, the integrals will require the use of the properties of Legendre polynomials.



$$\vec{r} \cdot \vec{R} = r R \cos\theta'$$

(6) This allows us to complete the ϕ' integral

$$\phi(\vec{r}) = \frac{R^2}{4\pi\epsilon_0} \int_0^\pi \sin\theta' d\theta' \int_0^{2\pi} d\phi' \, \frac{\vec{P}_0 \cdot \left\{ \hat{i} \sin\theta' \cos\phi' + \hat{j} \sin\theta' \sin\phi' + \hat{k} \cos\theta' \hat{k} \right\}}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$\int_0^{2\pi} d\phi' \cos\phi' = 0$$

$$\int_0^{2\pi} d\phi' \sin\phi' = 0$$

$$= \frac{R^2}{4\pi\epsilon_0} (\vec{P}_0 \cdot \hat{k}) \, 2\pi \int_0^\pi \sin\theta' d\theta' \, \frac{\cos\theta'}{\sqrt{r^2 + R^2 - 2rR\cos\theta'}}$$

$$= \frac{2\pi R^2}{4\pi\epsilon_0} (\vec{P}_0 \cdot \hat{k}) \int_{-1}^{+1} dt \, \frac{t}{\sqrt{r^2 + R^2 - 2rRt}}$$

7 <u>Homework</u> :

Show that, (using substitution $r^2+R^2-2rRt = y$,)

(i) $$\int_{-1}^{+1} dt \, \frac{1}{\sqrt{r^2+R^2-2rRt}} =$$

(ii) $$\int_{-1}^{+1} dt \, \frac{t}{\sqrt{r^2+R^2-2rRt}} = \begin{cases} \frac{2}{3}\frac{r}{R^2}, & r<R, \\ \frac{2}{3}\frac{R}{r^2}, & R<r \end{cases}$$

$$= \frac{2}{3}\frac{r_<}{r_>^2}$$

8 Using 7 in 6

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\vec{P}_0 \cdot \hat{k}\right) \begin{cases} \frac{4\pi}{3} r, & r<R, \\ \left(\frac{4\pi}{3}R^3\right)\frac{1}{r^2}, & R<r, \end{cases}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\vec{P}_0 \cdot \hat{r}\right) \begin{cases} \left(\frac{4\pi}{3}r^3\right)\frac{1}{r^2}, & r<R, \\ \left(\frac{4}{3}\pi R^3\right)\frac{1}{r^2}, & R<r. \end{cases}$$

(replaced $\hat{k}$ with $\hat{r}$, as per our choice in 5.)

(9) The electric field is obtained using

$$\vec{E} = - \vec{\nabla} \phi$$

$$= \frac{1}{4\pi \epsilon_0} \begin{cases} \frac{4\pi}{3} (-1)\, \vec{\nabla} \left[\vec{P}_0 \cdot \vec{r} \right], & r < R, \\ \frac{4\pi}{3} (-1)\, R^3\, \vec{\nabla} \left(\frac{\vec{P}_0 \cdot \vec{r}}{r^3} \right), & R < r, \end{cases}$$

$$= \frac{1}{4\pi \epsilon_0} \begin{cases} - \frac{4\pi}{3} \vec{P}_0, & r < R, \\ \left(\frac{4\pi}{3} R^3 \right) \frac{1}{r^3} \left[3 (\vec{P}_0 \cdot \hat{r})\, \hat{r} - \vec{P}_0 \right], & R < r. \end{cases}$$

Thus, outside the sphere the electric field is that of a point dipole. Inside the sphere it is a constant. That is, the charge accumulated on the surface is such that it produces uniform $\vec{E}$, which simulates a dipole.