Midterm Exam No. 01 (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Date: 2014 Sep 29

Solutions

1. **(20 points.)** Evaluate the integral

$$\int_{-1}^{1} dx\, \delta(1-2x)\Big[8x^2+2x-1\Big]. \tag{1}$$

2. **(20 points.)** A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density

$$\rho(\mathbf{r},t) = -\mathbf{p}\cdot\boldsymbol{\nabla}\delta^{(3)}(\mathbf{r}-\mathbf{r}_0). \tag{2}$$

Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r},t)$.

3. **(60 points.)** Consider circularly polarized light of infinite extent with fields given by

$$\mathbf{E} = cB_0\big[\hat{\mathbf{x}}\cos(kz-\omega t)+\hat{\mathbf{y}}\sin(kz-\omega t)\big], \tag{3}$$

$$\mathbf{B} = B_0\big[-\hat{\mathbf{x}}\sin(kz-\omega t)+\hat{\mathbf{y}}\cos(kz-\omega t)\big]. \tag{4}$$

(a) A plane wave is characterized by $\varepsilon_0 E^2 = \mu_0 H^2$ and $\mathbf{E}\cdot\mathbf{H}=0$. Does the above configuration satisfy the characteristics of a plane wave?

(b) Evaluate the electromagnetic energy density for the above configuration to be

$$U = \varepsilon_0 c^2 B_0^2. \tag{5}$$

(c) Evaluate the angular momentum density to be

$$\mathbf{L} = \mathbf{r}\times(\mathbf{D}\times\mathbf{B}) = \frac{U}{c}\,\mathbf{r}\times\hat{\mathbf{z}}. \tag{6}$$

(d) Determine the angular momentum flux tensor, along $\hat{\mathbf{z}}$,

$$\hat{\mathbf{z}}\cdot\mathbf{K} = -\hat{\mathbf{z}}\cdot(\mathbf{T}\times\mathbf{r}) = ?, \tag{7}$$

where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the circularly polarized light to be incident on the side $z = z_a - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z_a + \delta$, conclude that

$$\tau = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \hat{\mathbf{z}} \cdot \mathbf{K}\big|_{z=z_a-\delta}. \tag{12}$$

Use Eq. (7) in Eq. (12) and calculate the total torque on the plate.

(g) Refer [1] and problem 7.27 in Ref. [2] for a complete analysis. (Will not be graded.)

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[1] Hans C. Ohanian, "What is spin?" American Journal of Physics **54**, 500–505 (1986).

[2] John D. Jackson, *Classical electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., 1999).

MT-01, prob 1

$$\delta(1-2x) = \delta\left(-2\left(x-\tfrac{1}{2}\right)\right) = \frac{1}{2}\,\delta\left(x-\tfrac{1}{2}\right)$$

$$\int_{-1}^{1} dx \;\delta(1-2x)\left[8x^2+2x-1\right]$$

$$= \int_{-1}^{1} dx \;\frac{1}{2}\,\delta\left(x-\tfrac{1}{2}\right)\left[8x^2+2x-1\right]$$

$$= \frac{1}{2}\left[8\left(\tfrac{1}{2}\right)^2 + 2\left(\tfrac{1}{2}\right) - 1\right]$$

$$= \frac{1}{2}\left[2+1-1\right] = 1$$

MT-01, prob 2

$$\vec{F} = \int d^3r \;\rho(\vec{r},t)\;\vec{E}(\vec{r},t)$$

$$= -\int d^3r \left[\vec{p}\cdot\vec{\nabla}\,\delta^{(3)}(\vec{r}-\vec{r}_0)\right]\vec{E}(\vec{r},t)$$

$$= \int d^3r \;\delta^{(3)}(\vec{r}-\vec{r}_0)\left[\vec{p}\cdot\vec{\nabla}\;\vec{E}(\vec{r},t)\right]$$

$$= \vec{p}\cdot\vec{\nabla}_{\vec{r}_0}\;\vec{E}(\vec{r}_0,t)$$

MT-01, prob 3

$$\vec{E} = c B_0 \left[\hat{x} \, \mathrm{Cos}(kz-\omega t) + \hat{y} \, \mathrm{Sin}(kz-\omega t) \right]$$

$$\vec{B} = B_0 \left[-\hat{x} \, \mathrm{Sin}(kz-\omega t) + \hat{y} \, \mathrm{Cos}(kz-\omega t) \right]$$

(a)

$$\vec{E} \cdot \vec{B} = c B_0^2 \left[-\mathrm{Cos}(kz-\omega t)\,\mathrm{Sin}(kz-\omega t) + \mathrm{Sin}(kz-\omega t)\,\mathrm{Cos}(kz-\omega t) \right]$$

$$= 0 .$$

$$\epsilon_0 E^2 = \epsilon_0 c^2 B_0^2 \left[\mathrm{Cos}^2(kz-\omega t) + \mathrm{Sin}^2(kz-\omega t) \right]$$

$$= \epsilon_0 c^2 B_0^2 .$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2} .$$

$$\mu_0 H^2 = \frac{1}{\mu_0} B^2 = \frac{1}{\mu_0} B_0^2 \left[\mathrm{Sin}^2(kz-\omega t) + \mathrm{Cos}^2(kz-\omega t) \right]$$

$$= \epsilon_0 c^2 B_0^2$$

Thus, $\vec{E}\cdot\vec{B} = 0$ and $\epsilon_0 E^2 = \mu_0 H^2$.

(b)

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0 H^2 .$$

$$= \epsilon_0 E^2 \quad \text{(using (a))}$$

$$= \epsilon_0 c^2 B_0^2 \quad \text{(using (a))}$$

$$\vec{E}\times\vec{B} = c B_0^2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \mathrm{Cos} & \mathrm{Sin} & 0 \\ -\mathrm{Sin} & \mathrm{Cos} & 0 \end{vmatrix}$$

$$= c B_0^2 \hat{z}$$

(c)

$$\vec{L} = \vec{r} \times (\vec{D}\times\vec{B})$$

$$= \epsilon_0 \, \vec{r}\times(\vec{E}\times\vec{B})$$

$$= \epsilon_0 c B_0^2 \, \vec{r}\times\hat{z}$$

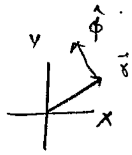
$$= \frac{U}{c}\, \vec{r}\times\hat{z} \quad \text{(using (b))}$$

(d)

$$\hat{z}\cdot\overleftrightarrow{K} = -\hat{z}\cdot\overleftrightarrow{T}\times\vec{r}$$

$$= -\hat{z}\cdot\left[\overleftrightarrow{1}\,U - \vec{D}\vec{E} - \vec{B}\vec{H}\right]\times\vec{r}$$

$$= -U\,\hat{z}\cdot\overleftrightarrow{1}\times\vec{r} + \underbrace{\hat{z}\cdot\vec{D}}_{=0}\,\vec{E}\times\vec{r} + \underbrace{\hat{z}\cdot\vec{B}}_{=0}\,\vec{H}\times\vec{r}$$



$$= -U\,\hat{z}\cdot\overleftrightarrow{1}\times\vec{r}$$

$$= -U\,\hat{z}\times\vec{r}$$

$$= U\,\vec{r}\times\hat{z} = -U r\,\hat{\phi}$$

(e)

$$\frac{\partial\vec{L}}{\partial t} + \vec{\nabla}\cdot\overleftrightarrow{K} + \vec{t} = 0$$

$$\underbrace{\int_V d^3r\,\frac{\partial\vec{L}}{\partial t}}_{\text{goes to zero for } \delta\to 0} + \int_V d^3r\,\vec{\nabla}\cdot\overleftrightarrow{K} + \underbrace{\int d^3r\,\vec{t}}_{\vec{\tau}} = 0$$

$$\vec{\tau} = -\int_V d^3r\,\vec{\nabla}\cdot\overleftrightarrow{K}$$

$$= -\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{z_a-\delta}^{z_a+\delta}dz\,\vec{\nabla}\cdot\overleftrightarrow{K}$$

(f)
$$\vec{\tau} = - \oint_S d\vec{a} \cdot \overleftrightarrow{K}$$

$$= - \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, (-\hat{z}) \cdot \overleftrightarrow{K}$$ (since only the $z = z_a - \delta$ surface contribute.)

$$= \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; \hat{z} \cdot \overleftrightarrow{K} \Big|_{z = z_a - \delta}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{+\infty} dy \; U \; \vec{r} \times \hat{z}$$

$$\vec{r} \times \hat{z} = -x\hat{y} + y\hat{x} = -\hat{\phi}$$

$$= \epsilon_0 c^2 B_0^2 \int_{-\infty}^{\infty} dx \int_{-\infty}^{+\infty} dy \; \vec{r} \times \hat{z}$$

$$= \epsilon_0 c^2 B_0^2 \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \, (-x\hat{y} + y\hat{x})$$ (odd functions).

$$= 0$$

(g) If the incident wave had a finite extent, it does exert a force.