Midterm Exam No. 01 (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Date: 2014 Sep 29

1. (**20 points.**) Evaluate the integral

$$\int_{-1}^{1} dx\, \delta(1-2x) \Big[8x^2 + 2x - 1\Big]. \tag{1}$$

2. (**20 points.**) A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density

$$\rho(\mathbf{r}, t) = -\mathbf{p} \cdot \boldsymbol{\nabla} \delta^{(3)}(\mathbf{r} - \mathbf{r}_0). \tag{2}$$

Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r}, t)$.

3. (**60 points.**) Consider circularly polarized light of infinite extent with fields given by

$$\mathbf{E} = cB_0 \big[\hat{\mathbf{x}} \cos(kz - \omega t) + \hat{\mathbf{y}} \sin(kz - \omega t)\big], \tag{3}$$

$$\mathbf{B} = B_0 \big[-\hat{\mathbf{x}} \sin(kz - \omega t) + \hat{\mathbf{y}} \cos(kz - \omega t)\big]. \tag{4}$$

(a) A plane wave is characterized by $\varepsilon_0 E^2 = \mu_0 H^2$ and $\mathbf{E} \cdot \mathbf{H} = 0$. Does the above configuration satisfy the characteristics of a plane wave?

(b) Evaluate the electromagnetic energy density for the above configuration to be

$$U = \varepsilon_0 c^2 B_0^2. \tag{5}$$

(c) Evaluate the angular momentum density to be

$$\mathbf{L} = \mathbf{r} \times (\mathbf{D} \times \mathbf{B}) = \frac{U}{c} \mathbf{r} \times \hat{\mathbf{z}}. \tag{6}$$

(d) Determine the angular momentum flux tensor, along $\hat{\mathbf{z}}$,

$$\hat{\mathbf{z}} \cdot \mathbf{K} = -\hat{\mathbf{z}} \cdot (\mathbf{T} \times \mathbf{r}) = ?, \tag{7}$$

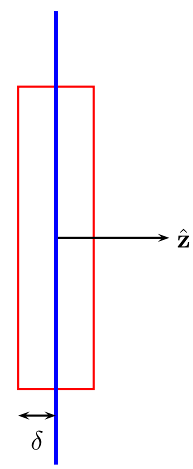

FIG. 1. A circularly polarized light incident on a screen.

where

$$\mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}). \tag{8}$$

(e) The above circularly polarized light is incident, in vacuum, on a perfectly absorbing flat screen. See Fig. 1. Without compromising generality we can choose the screen at $z = z_a$. Starting with the statement of conservation of angular momentum,

$$\frac{\partial \mathbf{L}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{K} + \mathbf{t} = 0, \tag{9}$$

integrate on the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Interpret the integral of torque density $\mathbf{t}$ as the total torque $\boldsymbol{\tau}$ on the plate. Further, note that the integral of angular momentum density $\mathbf{L}$ goes to zero for infinitely small δ. Thus, obtain

$$\boldsymbol{\tau} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz\, \boldsymbol{\nabla} \cdot \mathbf{K}. \tag{10}$$

(f) Use divergence theorem to conclude

$$\boldsymbol{\tau} = -\oint d\mathbf{a} \cdot \mathbf{K}, \tag{11}$$

where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the circularly polarized light to be incident on the side $z = z_a - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z_a + \delta$, conclude that

$$\boldsymbol{\tau} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \hat{\mathbf{z}} \cdot \mathbf{K}\big|_{z=z_a-\delta}. \tag{12}$$

Use Eq. (7) in Eq. (12) and calculate the total torque on the plate.

(g) Refer [1] and problem 7.27 in Ref. [2] for a complete analysis. (Will not be graded.)

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[1] Hans C. Ohanian, "What is spin?" American Journal of Physics **54**, 500–505 (1986).

[2] John D. Jackson, *Classical electrodynamics*, 3rd ed. (John Wiley & Sons, Inc., 1999).