Solutions

Midterm Exam No. 02 (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Date: 2014 Oct 28

1. **(25 points.)** Evaluate
$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^3}\right), \tag{1}$$
everywhere in space, including $\mathbf{r} = 0$.
Hint: Check your answer for consistency by using divergence theorem.

2. **(25 points.)** The response of a material to an electric field, in a particular model, is described by the susceptibility function
$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma}, \tag{2}$$
where ω_p, ω_0, and γ are material dependent parameters, and ω is the frequency of oscillation of the electric field.
   (a) $[\mathrm{Re}\chi(\omega)]$ is a measure of the square of the refractive index. Plot $[\mathrm{Re}\chi(\omega)]$ as a function of ω.
   (b) $[\mathrm{Im}\chi(\omega)]$ is a measure of absorption of light. Plot $[\mathrm{Im}\chi(\omega)]$ as a function of ω.

3. **(25 points.)** A permanently polarized sphere of radius R is described by the polarization vector
$$\mathbf{P}(\mathbf{r}) = \alpha r^2 \,\hat{\mathbf{r}}\, \theta(R - r). \tag{3}$$
Find the effective charge density by calculating $-\boldsymbol{\nabla} \cdot \mathbf{P}$. In particular, you should obtain two terms, one containing $\theta(R - r)$ that is interpreted as a volume charge density, and another containing $\delta(R - r)$ that can be interpreted as a surface charge density.

4. **(25 points.)** A particle of mass m and charge q moving in a uniform magnetic field $\mathbf{B}$ experiences a velocity dependent force $\mathbf{F}$ given by the expression
$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \tag{4}$$
where $\mathbf{v}(t) = d\mathbf{x}/dt$ is the velocity of the particle in terms of its position $\mathbf{x}(t)$. Choose the magnetic field to be along the positive z direction, $\mathbf{B} = B\hat{\mathbf{z}}$.
   (a) Using initial conditions $\mathbf{v}(0) = 0\,\hat{\mathbf{x}} + v_0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$ and $\mathbf{x}(0) = 0\,\hat{\mathbf{x}} + 0\,\hat{\mathbf{y}} + 0\,\hat{\mathbf{z}}$, solve the differential equation in Eq. (4) to find the position $\mathbf{x}(t)$ and velocity $\mathbf{v}(t)$ as a function of time.

MT-02, prob 1

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = \frac{(\vec{\nabla} \cdot \vec{r})}{r^3} + \vec{r} \cdot \vec{\nabla} \frac{1}{r^3}$$

$\vec{\nabla} r = \hat{r}$

$\vec{\nabla} \cdot \vec{r} = 3$

$$= \frac{3}{r^3} + \vec{r} \cdot \frac{(-3)}{r^4} \vec{\nabla} r$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0 \quad \text{if} \quad r \neq 0$$

If $r = 0$, $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = \infty$

Thus, it seems

$$\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = c\, \delta^{(3)}(\vec{r})$$

To find the constant we use the divergence theorem

$$\int_V d^3r\, \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = c \int_V d^3r\, \delta^{(3)}(\vec{r})$$

$$\int d\vec{a} \cdot \frac{\vec{r}}{r^3} = c$$

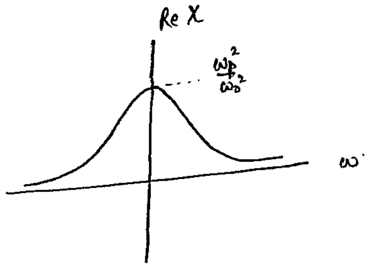
$$4\pi = c$$

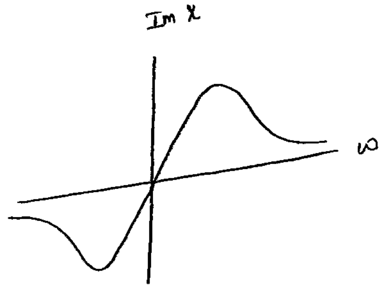
Thus, $\vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = 4\pi\, \delta^{(3)}(\vec{r})$

MT02, prob 2

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma} = \frac{\omega_p^2(\omega_0^2 + i\omega\gamma)}{\omega_0^4 + \omega^2\gamma^2}.$$

$$\mathrm{Re}[\chi(\omega)] = \frac{\omega_p^2\,\omega_0^2}{\omega_0^4 + \omega^2\gamma^2} \qquad \mathrm{Im}[\chi(\omega)] = \frac{\omega_p^2\,\gamma\,\omega}{\omega_0^4 + \omega^2\gamma^2}.$$





MT02, prob 3

$$\vec{P}(\vec{r}) = \alpha r^2\,\hat{r}\,\theta(R-r)$$

$$\begin{aligned}
\rho_{\text{eff}}(\vec{r}) &= -\vec{\nabla}\cdot\vec{P} \\
&= -\vec{\nabla}\cdot\left[\alpha r^2\,\hat{r}\,\theta(R-r)\right] \\
&= -\alpha\left(\vec{\nabla}\cdot r\,\vec{r}\right)\theta(R-r) - \alpha r^2\,\hat{r}\cdot\vec{\nabla}\,\theta(R-r) \\
&= -\alpha\left(\{\vec{\nabla} r\}\cdot\vec{r} + r\{\vec{\nabla}\cdot\vec{r}\}\right)\theta(R-r) - \alpha r^2\,\hat{r}\cdot(\vec{\nabla} r)\frac{\partial}{\partial r}\theta(R-r) \\
&= -\alpha\left(\hat{r}\cdot\vec{r} + r\,3\right)\theta(R-r) + \alpha r^2\,\hat{r}\cdot\hat{r}\,\delta(R-r) \\
&= -4\alpha r\,\theta(R-r) + \alpha r^2\,\delta(R-r)
\end{aligned}$$

MT02, prob 4

(a) $$\frac{d\vec{v}}{dt} = \frac{q}{m}\,\vec{v}\times\vec{B}$$

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ V_x & V_y & V_z \\ 0 & 0 & B \end{vmatrix} = B V_y\,\hat{i} - B V_x\,\hat{j} + 0\,\hat{k}.$$

$$\Rightarrow \frac{dV_x}{dt} = \frac{qB}{m} V_y \quad \text{— (i)}$$

$$\frac{dV_y}{dt} = -\frac{qB}{m} V_x \quad \text{— (ii)}$$

$$\frac{dV_z}{dt} = 0 \quad \text{— (iii)}$$

Using (i) and (ii)

$$\frac{d^2 V_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 V_x \quad \text{— (iv)}$$

$$\frac{d^2 V_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 V_y \quad \text{— (v)}$$

$$\omega_p = \frac{qB}{m}$$

Solution:
$$V_x(t) = A\sin\omega_p t + B\cos\omega_p t$$
$$V_y(t) = A\cos\omega_p t - B\sin\omega_p t$$
$$V_z(t) = C$$

Initial condition: $\vec{V}(0) = 0\,\hat{x} + V_0\,\hat{y} + 0\,\hat{z}.$

$$\Rightarrow C = 0, \quad A = V_0, \quad B = 0$$

Thus,
$$V_x(t) = V_0 \sin\omega_p t$$
$$V_y(t) = V_0 \cos\omega_p t$$
$$V_z(t) = 0$$

$$\frac{dx}{dt} = V_0 \sin \omega_p t \quad \Rightarrow \quad x(t) = -\frac{V_0}{\omega_p} \cos \omega_p t + x_0$$

$$\frac{dy}{dt} = V_0 \cos \omega_p t \quad \Rightarrow \quad y(t) = \frac{V_0}{\omega_p} \sin \omega_p t + y_0$$

$$\frac{dz}{dt} = 0 \quad \Rightarrow \quad z(t) = z_0$$

Initial condition : $\vec{X}(0) = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$.

$$\Rightarrow \quad x_0 = \frac{V_0}{\omega_p}, \quad y_0 = 0, \quad z_0 = 0$$

Thus,

$$x(t) = -\frac{V_0}{\omega_p} \cos \omega_p t + \frac{V_0}{\omega_p}$$

$$y(t) = \frac{V_0}{\omega_p} \sin \omega_p t$$

$$z(t) = 0$$

(b) $$\left(x(t) - \frac{V_0}{\omega_p}\right)^2 + y(t)^2 = \left(\frac{V_0}{\omega_p}\right)^2,$$

which is an equation of a circle.

Radius $= \frac{V_0}{\omega_p}$

Center : $\frac{V_0}{\omega_p}\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$

(c) $\omega_p = \frac{qB}{m}$, which is independent of the initial conditions.