Solutions

Final Exam (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Date: 2014 Dec 10

1. **(20 points.)** A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density

$$\rho(\mathbf{r}, t) = -\mathbf{p} \cdot \boldsymbol{\nabla} \delta^{(3)}(\mathbf{r} - \mathbf{r}_0). \tag{1}$$

Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r}, t)$.

2. **(20 points.)** Evaluate the integral

$$\int_{-1}^{1} dx\, \delta(3 - 2x) \Big[8x^2 + 2x - 1\Big]. \tag{2}$$

Caution: Notice and take into account the limits of integration.

3. **(20 points.)** A plane wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{3}$$
$$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{4}$$

where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. Assume no charges or currents.

(a) Using Maxwell's equations evaluate

$$\mathbf{k} \cdot \mathbf{E}, \quad \mathbf{k} \cdot \mathbf{B}, \quad \mathbf{k} \times \mathbf{E}, \quad \text{and} \quad \mathbf{k} \times \mathbf{B}. \tag{5}$$

(b) Further, using Eqs. (5), derive the relations

$$ck = \omega, \quad \hat{\mathbf{k}} \times \mathbf{e}_0 = c\mathbf{b}_0, \quad c\hat{\mathbf{k}} \times \mathbf{b}_0 = -\mathbf{e}_0, \quad \text{and} \quad cb_0 = e_0, \tag{6}$$

where $\varepsilon_0 \mu_0 = 1/c^2$.

(c) Evaluate the energy density

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \tag{7}$$

and the momentum density

$$\mathbf{G} = \varepsilon_0 \mathbf{E} \times \mathbf{B}. \tag{8}$$

Then, determine the ratio U/G.

Final Exam, prob. 1

$$\vec{F} = \int d^3r \; \rho \; E$$

$$= \int d^3r \left[- \vec{p} \cdot \vec{\nabla} \, \delta^{(3)}(\vec{r} - \vec{r}_0) \right] \vec{E}$$

$$= \int d^3r \; \delta^{(3)}(\vec{r} - \vec{r}_0) \; (+\vec{p} \cdot \vec{\nabla}) \; \vec{E} \quad + \underbrace{\text{surface term}}_{=0}$$

$$= + \vec{p} \cdot \vec{\nabla} \vec{E}$$

Final Exam, prob. 2

$$\int_{-1}^{+1} dx \; \delta(3-2x) \left[8x^2 + 2x - 1 \right] = 0$$

because $\frac{3}{2}$ is outside the domain of integration.

Final Exam, prob. 3

(a) $\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \vec{\nabla} \cdot \left[\vec{e}_0 \, e^{i\vec{k}\cdot\vec{r} - i\omega t} \right] = 0 \Rightarrow \vec{k} \cdot \vec{E} = 0$

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{\nabla} \cdot \left[\vec{b}_0 \, e^{i\vec{k}\cdot\vec{r} - i\omega t} \right] = 0 \Rightarrow \vec{k} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \vec{\nabla} \times \left[\vec{e}_0 \, e^{i\vec{k}\cdot\vec{r} - i\omega t} \right] = -\frac{\partial}{\partial t} \left[\vec{b}_0 \, e^{i\vec{k}\cdot\vec{r} - i\omega t} \right]$

$\Rightarrow i\vec{k} \times \vec{E} = i\omega \vec{B}$

$\Rightarrow \vec{k} \times \vec{E} = \omega \vec{B}$

$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial E}{\partial t} \Rightarrow i\vec{k} \times \vec{B} = -i\omega \, \mu_0 \epsilon_0 \vec{E}$

$\Rightarrow \vec{k} \times \vec{B} = -\frac{\omega}{c^2} \vec{E}$

(b)

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\vec{k} \times (\vec{k} \times \vec{E}) = \omega\, \vec{k} \times \vec{B}$$

$$\vec{k}\, \underbrace{(\vec{k} \cdot \vec{E})}_{=0} - k^2 \vec{E} = \omega \left(-\frac{\omega}{c^2}\right) \vec{E}$$

$$k^2 \vec{E} = \frac{\omega^2}{c^2} \vec{E} \quad \Rightarrow \quad kc = \omega .$$

$$\vec{k} \times \vec{B} = -\frac{\omega}{c^2} \vec{E}$$

$$\vec{k} \times \vec{b}_0 = -\frac{\omega}{c^2} \vec{e}_0$$

$$c\, \hat{k} \times \vec{b}_0 = -\vec{e}_0$$

$$\vec{k} \times \vec{E} = \omega \vec{B}$$

$$\vec{k} \times \vec{e}_0 = \omega \vec{b}_0$$

$$\hat{k} \times \vec{e}_0 = \frac{\omega}{k} \vec{b}_0 = c \vec{b}_0$$

$$\begin{aligned} c^2 b_0^2 &= c^2\, \vec{b}_0 \cdot \vec{b}_0 \\ &= \hat{k} \times \vec{e}_0 \cdot \hat{k} \times \hat{e}_0 \\ &= (\hat{k} \cdot \hat{k})(\vec{e}_0 \cdot \vec{e}_0) - \underbrace{(\hat{k} \cdot \vec{e}_0)^2}_{=0} \\ &= e_0^2 \end{aligned}$$

$$\Rightarrow \quad c\, b_0 = e_0$$

(c)

$$\begin{aligned} U &= \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \\ &= \frac{1}{2} \epsilon_0 e_0^2\, e^{2i\vec{k}\cdot\vec{r} - 2i\omega t} + \frac{1}{2\mu_0} b_0^2\, e^{2i\vec{k}\cdot\vec{r} - 2i\omega t} \\ &= \epsilon_0 e_0^2\, e^{2i\vec{k}\cdot\vec{r} - 2i\omega t} \end{aligned}$$

$$\frac{b_0^2}{\mu_0} = \epsilon_0 \frac{b_0^2}{\epsilon_0 \mu_0} = \epsilon_0 c^2 b_0^2 = \epsilon_0 e_0^2$$

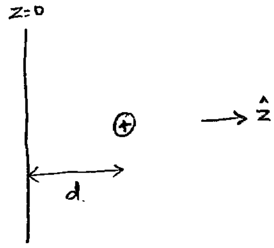
$$\begin{aligned} \vec{G} &= \epsilon_0\, \vec{e}_0 \times \vec{b}_0\, e^{2i\vec{k}\cdot\vec{r} - 2i\omega t} \\ &= \frac{1}{c} \hat{k}\, \epsilon_0 e_0^2\, e^{2i\vec{k}\cdot\vec{r} - 2i\omega t} . \end{aligned}$$

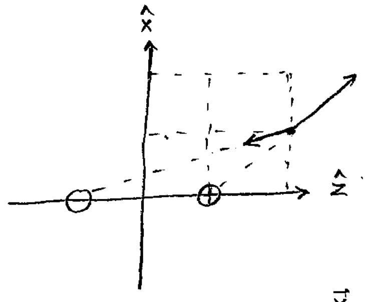
$$\begin{aligned} \vec{e}_0 \times \vec{b}_0 &= \frac{1}{c} \vec{e}_0 \times (\hat{k} \times \vec{e}_0) \\ &= \frac{1}{c} \hat{k}\, e_0^2 - \frac{1}{c} \vec{e}_0 \underbrace{(\hat{k} \cdot \vec{e}_0)}_{=0} \\ &= \frac{1}{c} \hat{k}\, e_0^2 . \end{aligned}$$

$$\Rightarrow \quad c\, |\vec{G}| = U$$

$$\frac{U}{G} = c .$$

Final Exam, prob. 4





$\vec{r} = d\,\hat{x} + 2d\,\hat{z}$

$\vec{r}_+ = 0\,\hat{x} + d\,\hat{z}$

$\vec{r}_- = 0\,\hat{x} - d\,\hat{z}$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_+}{|\vec{r}-\vec{r}_+|^3} - \frac{q}{4\pi\epsilon_0}\frac{\vec{r}-\vec{r}_-}{|\vec{r}-\vec{r}_-|^3}$$

$\vec{r}-\vec{r}_- = d\,\hat{x} + 3d\,\hat{z}$

$\vec{r}-\vec{r}_+ = d\,\hat{x} + d\,\hat{z}$

$|\vec{r}-\vec{r}_-| = \sqrt{10}\,d$

$|\vec{r}-\vec{r}_+| = \sqrt{2}\,d$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{d\,\hat{x}+d\,\hat{z}}{(\sqrt{2}\,d)^3} - \frac{q}{4\pi\epsilon_0}\frac{d\,\hat{x}+3d\,\hat{z}}{(\sqrt{10}\,d)^3}$$

$$= \frac{q}{4\pi\epsilon_0}\frac{1}{d^2}\left[\left(\frac{1}{2\sqrt{2}} - \frac{1}{10\sqrt{10}}\right)\hat{x} + \left(\frac{1}{2\sqrt{2}} - \frac{3}{10\sqrt{10}}\right)\hat{z}\right]$$

Final Exam, prob. 5

$$K_m\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right] I_m = 0$$

$$I_m\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right] K_m = 0$$

Subtracting the two equations we have.

$$I_m \frac{1}{t}\frac{d}{dt}t\frac{d}{dt}K_m - K_m\frac{1}{t}\frac{d}{dt}t\frac{d}{dt}I_m = 0$$

$$I_m \frac{1}{t}\frac{d}{dt}(t\,K_m') - K_m\frac{1}{t}\frac{d}{dt}(t\,I_m') = 0$$

$$I_m \frac{d}{dt}\left(t K_m'\right) - K_m \frac{d}{dt}\left(t I_m'\right) = 0$$

$$\frac{d}{dt}\left(I_m \, t \, K_m'\right) - I_m' \, t \, K_m' - \frac{d}{dt}\left(K_m \, t \, I_m'\right) + K_m' \, t \, I_m' = 0$$

$$\frac{d}{dt}\left(I_m \, t \, K_m' - K_m \, t \, I_m'\right) = 0$$

$$t\left(I_m K_m' - K_m I_m'\right) = -C$$

$$I_m K_m' - K_m I_m' = -\frac{C}{t}.$$

For $t \to \infty$

$$I_m K_m' - K_m I_m' = \frac{1}{\sqrt{2\pi}} \frac{e^{t}}{\sqrt{t}} \frac{d}{dt}\left(\sqrt{\frac{\pi}{2}} \frac{e^{-t}}{\sqrt{t}}\right) - \sqrt{\frac{\pi}{2}} \frac{e^{-t}}{\sqrt{t}} \frac{d}{dt}\left(\frac{1}{\sqrt{2\pi}} \frac{e^{t}}{\sqrt{t}}\right)$$

$$= \frac{1}{2} \frac{e^{t}}{\sqrt{t}} \frac{d}{dt}\left(\frac{e^{-t}}{\sqrt{t}}\right) - \frac{1}{2} \frac{e^{-t}}{\sqrt{t}} \frac{d}{dt}\left(\frac{e^{t}}{\sqrt{t}}\right)$$

$$= -\frac{1}{2t} - \frac{1}{2t} + O\left(t^{-\frac{3}{2}}\right)$$

$$= -\frac{1}{t}$$

$$\Rightarrow \quad C = 1.$$