Final Exam (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Date: 2014 Dec 10

1. (**20 points.**) A point dipole $\mathbf{p}$, stationary at position $\mathbf{r}_0$, is described by the charge density
$$\rho(\mathbf{r}, t) = -\mathbf{p} \cdot \boldsymbol{\nabla} \delta^{(3)}(\mathbf{r} - \mathbf{r}_0). \tag{1}$$
Determine the force on the point dipole in an electric field $\mathbf{E}(\mathbf{r}, t)$.

2. (**20 points.**) Evaluate the integral
$$\int_{-1}^{1} dx\, \delta(3 - 2x) \Big[8x^2 + 2x - 1 \Big]. \tag{2}$$
Caution: Notice and take into account the limits of integration.

3. (**20 points.**) A plane wave is described by electric and magnetic fields of the form
$$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{3}$$
$$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{4}$$
where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. Assume no charges or currents.

   (a) Using Maxwell's equations evaluate
   $$\mathbf{k} \cdot \mathbf{E}, \quad \mathbf{k} \cdot \mathbf{B}, \quad \mathbf{k} \times \mathbf{E}, \quad \text{and} \quad \mathbf{k} \times \mathbf{B}. \tag{5}$$

   (b) Further, using Eqs. (5), derive the relations
   $$ck = \omega, \quad \hat{\mathbf{k}} \times \mathbf{e}_0 = c\mathbf{b}_0, \quad c\hat{\mathbf{k}} \times \mathbf{b}_0 = -\mathbf{e}_0, \quad \text{and} \quad cb_0 = e_0, \tag{6}$$
   where $\varepsilon_0 \mu_0 = 1/c^2$.

   (c) Evaluate the energy density
   $$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \tag{7}$$
   and the momentum density
   $$\mathbf{G} = \varepsilon_0 \mathbf{E} \times \mathbf{B}. \tag{8}$$
   Then, determine the ratio U/G.

4. (**20 points.**) A perfectly conducting plate is placed at $z = 0$ plane. A positive charge q is placed at $\mathbf{r} = d\,\hat{\mathbf{z}}$. Using method of images determine the direction and magnitude of the electric field at the point $\mathbf{r} = d\,\hat{\mathbf{x}} + 2d\,\hat{\mathbf{z}}$.

5. (**20 points.**) The modified Bessel functions, $I_m(t)$ and $K_m(t)$, satisfy the differential equation

$$\left[-\frac{1}{t}\frac{d}{dt}t\frac{d}{dt} + \frac{m^2}{t^2} + 1\right]\left\{\begin{array}{c} I_m(t) \\ K_m(t) \end{array}\right\} = 0. \tag{9}$$

Derive the identity, for the Wronskian, (upto a constant C)

$$I_m(t)K_m'(t) - K_m(t)I_m'(t) = -\frac{C}{t}, \tag{10}$$

where

$$I_m'(t) \equiv \frac{d}{dt}I_m(t) \qquad \text{and} \qquad K_m'(t) \equiv \frac{d}{dt}K_m(t). \tag{11}$$

Further, determine the value of the constant C on the right hand side of Eq. (10) using the asymptotic forms for the modified Bessel functions:

$$I_m(t) \xrightarrow{t\gg 1} \frac{1}{\sqrt{2\pi}}\frac{e^t}{\sqrt{t}}, \tag{12}$$

$$K_m(t) \xrightarrow{t\gg 1} \sqrt{\frac{\pi}{2}}\frac{e^{-t}}{\sqrt{t}}. \tag{13}$$