Homework No. 03 (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Due date: Tuesday, 2014 Sep 16, 4.00pm

1. (Ref. Milton's lecture notes.) A plane wave is described by electric and magnetic fields of the form

$$\mathbf{E} = \mathbf{e}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{1}$$
$$\mathbf{B} = \mathbf{b}_0 e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}, \tag{2}$$

where $\mathbf{e}_0$ and $\mathbf{b}_0$ are constants. From Maxwell's equations in free space (no charges or currents)

   (a) Determine the relation between $\mathbf{e}_0$, $\mathbf{b}_0$, and $\mathbf{k}$.

   (b) Determine the relation between ω and $\mathbf{k}$.

   (c) Verify the statement of conservation of energy for a plane wave.

   (d) Verify the statement of conservation of momentum for a plane wave.

2. A plane wave is incident, in vacuum, on a perfectly absorbing flat screen.

   (a) Without compromising generality we can choose the screen at $z = z_a$. Starting with the statement of conservation of linear momentum,

$$\frac{\partial \mathbf{G}}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{T} + \mathbf{f} = 0, \tag{3}$$

   integrate on the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Interpret the integral of force density $\mathbf{f}$ as the total force, $\mathbf{F}$, on the plate. Further, note that the integral of momentum density $\mathbf{G}$ goes to zero for infinitely small δ. Thus, obtain

$$\mathbf{F} = -\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{z_a-\delta}^{z_a+\delta} dz\, \boldsymbol{\nabla}\cdot\mathbf{T}. \tag{4}$$

   (b) Use divergence theorem to conclude

$$\mathbf{F} = -\oint d\mathbf{a}\cdot\mathbf{T}, \tag{5}$$

   where the closed surface encloses the volume between $z = z_a - \delta$ and $z = z_a + \delta$ for infinitely small $\delta > 0$. Choose the plane wave to be incident on the side $z = z - \delta$ of the plate, and assuming $\mathbf{E} = 0$ and $\mathbf{B} = 0$ on the side $z = z + \delta$, conclude that

$$\frac{\mathbf{F}}{A} = \hat{\mathbf{z}}\cdot\mathbf{T}\big|_{z=z_a-\delta}, \tag{6}$$

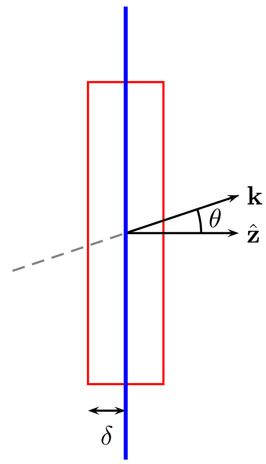

Figure 1: A plane wave with direction of propagation $\mathbf{k}$ incident on a screen.

where A is the total area of the screen. The electromagnetic stress tensor $\mathbf{T}$ in these expressions is given by

$$\mathbf{T} = \mathbf{1}U - (\mathbf{DE} + \mathbf{BH}), \tag{7}$$

where U is the electromagnetic energy density,

$$U = \frac{1}{2}(\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}). \tag{8}$$

(c) For the particular case when the plane wave is incident normally on the screen ($\theta = 0$ in Fig. 1) calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{z}}}{A}. \tag{9}$$

Express the answer in terms of U using the properties of a plane wave: $\mathbf{k} \cdot \mathbf{E} = 0, \mathbf{k} \cdot \mathbf{B} = 0, \mathbf{E} \cdot \mathbf{B} = 0, |\mathbf{E}| = c|\mathbf{B}|$, and $kc = \omega$.

(d) Consider the case when the plane wave is incident obliquely on the screen such that $\hat{\mathbf{k}} \cdot \hat{\mathbf{z}} = \cos\theta$ and $\mathbf{H} \cdot \hat{\mathbf{z}} = 0$. Calculate the force per unit area in the direction normal to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{z}}}{A}, \tag{10}$$

and the force per unit area tangential to the screen by evaluating

$$\frac{\mathbf{F} \cdot \hat{\mathbf{x}}}{A}. \tag{11}$$

Express the answer in terms of U and θ using the properties of a plane wave.