Homework No. 06 (Fall 2014)

PHYS 520A: Electromagnetic Theory I

Due date: Tuesday, 2014 Nov 4, 4.00pm

1. Consider the Green's function equation

$$-\left(\frac{d^2}{dt^2}+\omega^2\right)G(t)=\delta(t). \tag{1}$$

   Verify, by substituting into Eq. (1), that

$$G(t)=-\frac{1}{\omega}\theta(t)\sin\omega(t), \tag{2}$$

   is a particular solution to the Green's function equation.

2. Find the solution to the differential equation

$$\left[-\frac{\partial}{\partial z}\varepsilon(z)\frac{\partial}{\partial z}+\varepsilon(z)k_\perp^2\right]g(z,z';k_\perp)=\delta(z-z') \tag{3}$$

   when

$$\varepsilon(z)=\begin{cases}\varepsilon_2 & z<a,\\ \varepsilon_1 & a<z.\end{cases} \tag{4}$$

   for the case $a<z'$. Look for solutions that is zero at $z=\pm\infty$.

3. The expression for the electric potential due to a point charge placed in between two perfectly conducting semi-infinite slabs described by

$$\varepsilon(z)=\begin{cases}\infty, & z<0,\\ \varepsilon_0, & 0<z<a,\\ \infty, & a<z,\end{cases} \tag{5}$$

   is given in terms of the reduced Green's function that satisfies the differential equation $(0<\{z,z'\}<a)$

$$\left[-\frac{\partial^2}{\partial z^2}+k^2\right]\varepsilon_0 g(z,z')=\delta(z-z') \tag{6}$$

   with boundary conditions requiring the reduced Green's function to vanish at $z=0$ and $z=a$.

(a) Construct the reduced Green's function in the form

$$g(z,z') = \begin{cases} A\sinh kz + B\cosh kz, & 0 < z < z' < a, \\ C\sinh kz + D\cosh kz, & 0 < z' < z < a, \end{cases} \tag{7}$$

and solve for the four coefficients, A, B, C, D, using the conditions

$$g(0,z') = 0, \tag{8a}$$

$$g(a,z') = 0, \tag{8b}$$

$$g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = 0, \tag{8c}$$

$$\varepsilon_0 \partial_z g(z,z')\Big|_{z=z'-\delta}^{z=z'+\delta} = -1. \tag{8d}$$

Hint: The hyperbolic functions here are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \qquad \text{and} \qquad \cosh x = \frac{1}{2}(e^x + e^{-x}). \tag{9}$$

(b) Take the limit $ka \to \infty$ in your solution above to obtain the reduced Green's function for a single perfectly conducting slab,

$$\lim_{ka\to\infty} g(z,z') = \frac{1}{\varepsilon_0}\frac{1}{2k}e^{-k|z-z'|} - \frac{1}{\varepsilon_0}\frac{1}{2k}e^{-k|z|}e^{-k|z'|}. \tag{10}$$

This should serve as a check for your solution to the reduced Green's function.

4. Consider a semi-infinite dielectric slab described by

$$\varepsilon(z) = \begin{cases} \varepsilon_2 & z < a, \\ \varepsilon_1 > \varepsilon_2 & a < z. \end{cases} \tag{11}$$

(a) Find the expression for the electric field due to a point charge q placed at $\mathbf{r}'$ (with $a < z'$).

(b) Investigate the continuity in the components of electric field found above at the interface by evaluating the following:

$$E_x(x,y,a+\delta) - E_x(x,y,a-\delta) = ?, \tag{12}$$

$$E_y(x,y,a+\delta) - E_y(x,y,a-\delta) = ?, \tag{13}$$

$$\varepsilon_1 E_z(x,y,a+\delta) - \varepsilon_2 E_z(x,y,a-\delta) = ?. \tag{14}$$