# Homework No. 04 (Spring 2015)

## PHYS 420: Electricity and Magnetism II

Due date: Monday, 2015 Mar 2, 4.30pm

1. (**40 points.**) Generate 3D plots of surface spherical harmonics $Y_{lm}(\theta, \phi)$ as a function of $\theta$ and $\phi$. In particular,

   (a) Plot $\text{Re}\big[Y_{73}(\theta, \phi)\big]$.

   (b) Plot $\text{Im}\big[Y_{73}(\theta, \phi)\big]$.

   (c) Plot $\text{Abs}\big[Y_{73}(\theta, \phi)\big]$.

   (d) Plot your favourite spherical harmonic, that is, choose a $l$ and $m$, and Re or Im or Abs.

   Hint: In Mathematica these plots are generated using the following commands:
   `SphericalPlot3D[Re[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ϕ,0,2 Pi}]`
   `SphericalPlot3D[Im[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ϕ,0,2 Pi}]`
   `SphericalPlot3D[Abs[SphericalHarmonicY[l,m,θ,ϕ]],{θ,0,Pi},{ϕ,0,2 Pi}]`
   Refer to diagrams in Wikipedia article on 'spherical harmonics' to see some visual representations of these functions.

2. (**20 points.**) Verify that the right hand side of

$$(-\mathbf{a} \cdot \boldsymbol{\nabla}) \frac{1}{r} = \frac{\mathbf{a} \cdot \mathbf{r}}{r^3} \tag{1}$$

   is a solution to Laplace's equation for $\mathbf{r} \neq 0$. Further, verify the relation

$$(\mathbf{a}_1 \cdot \boldsymbol{\nabla})(\mathbf{a}_2 \cdot \boldsymbol{\nabla}) \frac{1}{r} = \frac{1}{r^5} \Big[ 3(\mathbf{a}_1 \cdot \mathbf{r})(\mathbf{a}_2 \cdot \mathbf{r}) - (\mathbf{a}_1 \cdot \mathbf{a}_2) r^2 \Big], \tag{2}$$

   which is also a solution to Laplace's equation for $\mathbf{r} \neq 0$, but need not be verified here.

3. (**10 points.**) The generating function for the spherical harmonics, $Y_{lm}(\theta, \phi)$, is

$$\frac{1}{l!} \left( \mathbf{a} \cdot \frac{\mathbf{r}}{r} \right)^l = \sum_{m=-l}^{l} \sqrt{\frac{4\pi}{2l+1}}\, Y_{lm}(\theta, \phi)\, \psi_{lm}, \tag{3}$$

   where the left hand side is expressed in terms of

$$\mathbf{r} = r(\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \tag{4}$$

$$\mathbf{a} = \frac{1}{2}(y_-^2 - y_+^2, -iy_-^2 - iy_+^2, 2y_- y_+), \tag{5}$$

and the right hand side consists of

$$\psi_{lm} = \frac{y_+^{l+m}}{\sqrt{(l+m)!}} \frac{y_-^{l-m}}{\sqrt{(l-m)!}} \tag{6}$$

and

$$Y_{lm}(\theta, \phi) = e^{im\phi} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}} \frac{1}{(\sin\theta)^m} \left( \frac{d}{d\cos\theta} \right)^{l-m} \frac{(\cos^2\theta - 1)^l}{2^l l!}. \tag{7}$$

Show that

$$\left( \mathbf{a} \cdot \frac{\mathbf{r}}{r} \right) \tag{8}$$

is unchanged by the substitution: $y_+ \leftrightarrow y_-$, $\theta \to -\theta$, $\phi \to -\phi$. Thus, show that

$$Y_{lm}(\theta, \phi) = Y_{l,-m}(-\theta, -\phi). \tag{9}$$

4. (**30 points.**) Write down the explicit forms of the spherical harmonics $Y_{lm}(\theta, \phi)$ for $l = 0, 1, 2$, by completing the $l - m$ differentiations in Eq. (7). Use the result in Eq. (9) to reduce the work by about half.

5. (**50 points.**) Legendre polynomials of order $l$ is given by (for $|t| < 1$)

$$P_l(t) = \left( \frac{d}{dt} \right)^l \frac{(t^2 - 1)^l}{2^l l!}. \tag{10}$$

   (a) Write down the explicit forms of the Legendre polynomials $P_l(t)$ for $l = 0, 1, 2, 3$, by completing the $l$ differentiations in Eq. (10).

   (b) Show that the spherical harmonics for $m = 0$ involves the Legendre polynomials,

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta). \tag{11}$$

6. (**40 points.**) Consider charge $Q$ uniformly distributed on a spherical shell of radius $R$.

   (a) Calculate the dipole moment of this charge distribution about the center of the shell.

   (b) Calculate the dipole moment of this charge distribution about the point $\mathbf{r}_0 = \frac{R}{2}\hat{\mathbf{k}}$.

   (c) What is the analog of the dipole moment in gravity?

   (d) Rate of change in quadruple moment of the source of gravity emits gravitational waves. Envisage (simplest possible) mass distributions that could emit gravitational waves.