

Homework No. 01 (Spring 2015)

PHYS 520B: Electromagnetic Theory

Due date: Monday, 2015 Feb 2, 4.30pm

1. Show that

$$\nabla \cdot \varepsilon(\rho) \nabla = \frac{1}{\rho} \frac{\partial}{\partial \rho} \varepsilon(\rho) \rho \frac{\partial}{\partial \rho} + \frac{\varepsilon(\rho)}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \varepsilon(\rho) \frac{\partial^2}{\partial z^2}, \quad (1)$$

where (ρ, ϕ, z) are the cylindrical coordinates.

2. Verify by substitution that

$$g_m(t, t') = I_m(t_{<}) K_m(t_{>}) \quad (2)$$

satisfies the differential equation

$$\left[-\frac{1}{t} \frac{\partial}{\partial t} t \frac{\partial}{\partial t} + \frac{m^2}{t^2} + 1 \right] g_m(t, t') = \frac{\delta(t - t')}{t}. \quad (3)$$

3. The cylindrical Green's function satisfies

$$\left[-\frac{1}{\rho} \frac{\partial}{\partial \rho} \varepsilon(\rho) \rho \frac{\partial}{\partial \rho} + \varepsilon(\rho) \frac{m^2}{\rho^2} + \varepsilon(\rho) k_z^2 \right] g_m(\rho, \rho'; k_z) = \frac{\delta(\rho - \rho')}{\rho}. \quad (4)$$

Consider a dielectric cylinder described by

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2 & \text{for } \rho < a, \\ \varepsilon_1 & \text{for } a < \rho. \end{cases} \quad (5)$$

- (a) Integrate Eq. (4) around $\rho = \rho'$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = 0, \quad (6a)$$

$$\varepsilon(\rho) \rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z) \Big|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1. \quad (6b)$$

- (b) Integrate Eq. (4) around $\rho = a$ to derive the continuity conditions:

$$g_m(\rho, \rho'; k_z) \Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0, \quad (7a)$$

$$\varepsilon(\rho) \rho \frac{\partial}{\partial \rho} g_m(\rho, \rho'; k_z) \Big|_{\rho=a-\delta}^{\rho=a+\delta} = 0. \quad (7b)$$

(c) For

$$\varepsilon(\rho) = \begin{cases} \varepsilon_2, & \rho < a, \\ \varepsilon_1, & a < \rho, \end{cases} \quad (8)$$

derive the solution

$$\underline{\rho' < a:}$$

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) I_m(k_z \rho') \frac{K_a K'_a}{\Delta}, & \rho, \rho' < a, \\ \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) I_m(k_z \rho') \frac{I'_a K_a}{\Delta}, & \rho' < a < \rho. \end{cases} \quad (9)$$

$$\underline{a < \rho':}$$

$$g_m(\rho, \rho'; k_z) = \begin{cases} \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) K_m(k_z \rho') \frac{I_a K'_a}{\Delta}, & \rho < a < \rho', \\ \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) K_m(k_z \rho') \frac{I_a I'_a}{\Delta}, & a < \rho, \rho'. \end{cases} \quad (10)$$

We used the definitions

$$\frac{1}{\Delta} = \frac{(\varepsilon_1 - \varepsilon_2)}{(\varepsilon_1 I_a K'_a - \varepsilon_2 K_a I'_a)}, \quad I_a \equiv I_m(k_z a), \quad K_a \equiv K_m(k_z a). \quad (11)$$

(d) Show that in the perfect conductor limit

Inside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_2} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_2} I_m(k_z \rho) I_m(k_z \rho') \frac{K_a}{I_a}. \quad (12)$$

Outside the cylinder

$$g_m(\rho, \rho'; k_z) = \frac{1}{\varepsilon_1} I_m(k_z \rho_{<}) K_m(k_z \rho_{>}) - \frac{1}{\varepsilon_1} K_m(k_z \rho) K_m(k_z \rho') \frac{I_a}{K_a}. \quad (13)$$